Criterion of existence of longitudinally inhomogeneous traveling waves in nonlinear electrodynamics

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The criterion referred to in the title is established solely on the basis of the behavior of the function 
\( \varepsilon(u)u^4 \), where \( u \) is the amplitude of the field in question and \( \varepsilon(u) \) is the nonlinear permittivity of the medium involved. The criterion is used to discuss the possibility of realization of one-dimensional longitudinally inhomogeneous waves under various conditions and, in particular, in nonlinear optics.

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In a dissipation-free medium, whose permittivity \( \varepsilon \) is a function of the field intensity \( |E|^2 \) at a given point, one-dimensional (plane) homogeneous traveling waves of constant frequency \( \omega_0 \) and amplitude \( \mu \) can exist in the same way as in a linear medium:

\[
E(x) = \text{const} - u, \quad E(x) = u \exp \left( \pm ik \sqrt{\varepsilon(u)} \right), \quad k = \omega_0 \varepsilon^{1/4},
\]  

(1)
where \( k = \omega/c \) and the amplitude-dependent phase velocity is constant. A basically new feature of the nonlinear case is the existence of longitudinally inhomogeneous traveling waves whose amplitude and velocity depend also on the coordinate: \( E(x) = u(x) \exp[i/k(x)dx] \), where \( u(x) \) and \( k(x) \) are real functions [the total field is \( \bar{E} = E \exp(-i\omega t) + \text{c.c.} \)]. The longitudinal aspect of the inhomogeneity is stressed in order to distinguish this case from the effects associated with the transverse inhomogeneity of a wave, such as self-focusing and self-bending. If the wave parameters vary along the propagation axis \( x \), the equation for the field is

\[
\frac{\partial E}{\partial x} + \frac{1}{2} \left[ E(x) E_\| - E_\perp^2 \right] = 0 \quad (\text{at } x = 0) \tag{2}
\]

In the case of "oblique" waves (for example, in the case when a field in a seminfinite nonlinear medium is excited by a wave incident from a nonlinear medium of permittivity \( \epsilon_d \) on a plane interface between the media and the angle of incidence is \( \varphi \), measured relative to the \( x \) axis which is normal to the boundary and to the polarization vector), Eq. (2) has to be modified by replacing \( E(u) \) with \( E_\| - \epsilon_d \sin^2 \varphi \).

A basically new feature is the appearance of purely traveling waves, because in other cases (when fluxes travel in opposite directions) there are always waves with a periodic structure of the amplitude in space, analogous to linear standing (or partly standing) waves. We shall define a purely traveling wave by the condition that the wave tends to become homogeneous at infinity where there are no radiation sources, i.e., in the limit \( x \rightarrow \infty \) we have

\[
E = \text{const} \geq 0; \quad k(x) = \text{const} = k_0 \quad \forall \left( u(x) \right) = k_0 > 0 \tag{3}
\]

where \( u \) and \( k \) are real quantities. It should be noted that Eq. (2) describes also the motion of a particle in a central field and then \( x \) is the time, \( u \) is the radius, \( k \) is the instantaneous angular velocity, and \( k_0 \epsilon_d \) is the central potential with the opposite sign. In this case, purely traveling waves are orbits which become circular for \( t \rightarrow \pm \infty \).

The waves and motion described by Eq. (2) are well known in mechanics and in nonlinear electrodynamics of plasmas [see, for example, Ref. 4, where a detailed classification of these waves—based on the integration of Eq. (2)—is given]. However, in the case of traveling waves (3) we still do not know the answer to a number of important and seemingly simple questions, the main of which can be formulated as follows: Is it possible to use directly the form of the function \( \epsilon(u) \) to determine the possibility of existence of spatially inhomogeneous traveling waves without integration of Eq. (2)? This is an important question because integration of Eq. (2) shows that in many typical types of nonlinearity of \( \epsilon(u) \) when the permittivity does not pass through zero [see, for example, Eq. (7) below], this equation admits the existence only of homogeneous traveling waves and not of longitudinally inhomogeneous waves. In particular, it is interesting that longitudinally inhomogeneous traveling waves have not yet been found in nonlinear optics if the same equation (2) is applied in the one-dimensional case. Therefore, the following question arises: Are longitudinally inhomogeneous traveling waves possible in optics at all and if the answer is in the affirmative, under what conditions?

We shall state and prove a comprehensive and single-valued criterion for the existence of longitudinally inhomogeneous traveling waves, which is based only on the behavior of the function \( F(u) = \epsilon(u) u_\| \) (the criterion is formulated without proof in Ref. 5).

1. Equation (2) admits a solution of the traveling wave type (3) if, and only if, there exists a range \( (U) \) of values \( u \) in which \( \epsilon(u) > 0 \) and if for any \( u \) in the range \( (U) \) there is a homogeneous traveling wave solution of the type described by Eq. (1). [This point is well-known and trivial, and is given here for the sake of completeness; it follows from an analysis of the asymptotic behavior of \( k \) in Eq. (3)].

2. If \( F(u) \) is continuous (or piecewise continuous) and is not falling within the range \( (U) \), then traveling wave solutions can only be homogeneous as in Eq. (1), i.e., longitudinally inhomogeneous traveling waves are impossible.

3. If \( F(u) \) is continuous in the range \( (U) \) and has a continuous or a piecewise-continuous derivative, solutions in the form of longitudinally inhomogeneous traveling waves are possible if, and only if, \( F(u) \) falls at least somewhere in \( (U) \), i.e., if there is an interval of monotonic fall in \( (U) \).

4. If the conditions in point 3 are satisfied and also if the derivative \( dF/du \) is finite in \( (U) \), Eq. (2) admits solutions in the form of longitudinally inhomogeneous traveling waves with a given wave amplitude at infinity \( u_\infty \), if and only if, \( u_\infty \) lies in one of the intervals \( (Y) \) where the function \( F(u) \) falls monotonically in \( (U) \).

The proof is based on an investigation of the value of one of the integrals of motion. Writing the field in Eq. (2) in the form \( E = u(x) \exp[i/k(x)dx] \), we obtain the first integral of motion of Eq. (2) \( u_1 = \text{const} \) (energy flux conservation) and the following equation \( u(x) \):

\[
\frac{d^2 u}{dx^2} + \frac{1}{2} \left[ \epsilon(u) - \frac{\epsilon'(u)}{\epsilon'} \right] u_1 = 0 \tag{2a}
\]

Bearing in mind that in the case of traveling waves we have \( k = k_0 \omega^2/\mu \), the first integral of Eq. (2a) satisfying the traveling wave condition (3) can be obtained in the form

\[
\left( \frac{du}{dx} \right)^2 = \frac{1}{k_0^2} \int_{u_\infty}^u \left[ F(u) - F(u_\infty) \right] \left( \frac{dF}{du} \right) \left( \frac{du}{dx} \right)^2 \tag{4}
\]

[the generalization of the laws of conservation to arbitrary field configurations and nonlinear media with \( \epsilon' = \epsilon(\mu) \) is made in Ref. 6].

The point 2 in the above criterion and the converse stated in point 3 follow from the fact that if we assume...
The existence of longitudinally inhomogeneous traveling waves, i.e., if \( u(x) = u_\infty \) in the case of a nonfalling function \( F(u) \), we find that the right-hand side of Eq. (4) is always nonpositive, whereas the left (with the exception of perhaps a few points) is always positive; thus, in this case we can only have homogeneous traveling waves \( u = u_\infty \). The direct statements in points 3 and 4 follow from the fact that if \( F(u) \) is not a nonfalling function in \( (0, U) \), then \( U \) should have such intervals \( (Y) \) in which \( df(du)/dX < 0 \) and where we can always select points \( u \) and \( u_\infty \) at which this derivative is finite and continuous; then, the right-hand side of Eq. (4) is positive and this equation has a solution in the form of longitudinally inhomogeneous traveling waves and it is easy to show that for \( x \to \infty \) (and \( u \to \infty \)) this solution is of the form \( u(x) = u_\infty + \Delta u + o(\Delta u) \), where

\[
\Delta u = \exp \left( -\int \frac{df}{du} \, dx + \text{const} \right) \frac{du_\infty}{u_\infty}.
\]

When the conditions stated in point 4 are satisfied, such solutions can exist for any point inside an interval \( (Y) \) and if this interval is closed, then this applies to its end points at which \( f(u_\infty) > 0 \) but \( df(du)/dX = 0 \). In the latter case if \( f''(u_\infty) = 0 \), then inhomogeneous longitudinal traveling waves in the limit \( x \to \infty \) have \( \Delta u \) described by

\[
\Delta u = -\frac{1}{2} \frac{d^2}{dX^2} f(u_\infty)(x + \text{const})^2.
\]

The converse of the conclusion in point 4 is proved in the same way as point 2.

We shall now consider some consequences in the above criterion.

1. If \( f(u) \) is a nonfalling function, then longitudinally inhomogeneous traveling waves are impossible. In the theory of nonlinear hysteretic reflection and refraction\(^7\) this means that in the case of rising \( f(u) \) we have only the choice between homogeneous traveling waves and a surface wave (corresponding to total reflection \( k = 0 \)) in a nonlinear medium.

2. If \( f(u) \) is of the form

\[
s(u) = \epsilon u(1 + u^2)/(1 + \beta u^2) \quad (\epsilon > 0, \beta > 0).
\]

which describes an extensive class of nonlinear mechanisms with saturation (both rising and falling right down to zero for \( \alpha = 0 \)), then in the case of a wave incident normally on an interface between media, we cannot have longitudinally inhomogeneous traveling waves. They can be obtained for special types of non-negative nonlinearities: for example, if \( f = \epsilon/(1 + \beta(u^2 + \cdots + u^{2n})) \), longitudinally homogeneous traveling waves are possible only if \( n > 3 \).

3. In the case of normal incidence of a wave and on condition that \( \Delta \zeta_0 \ll \zeta(0) \) (nonlinear optics), longitudinally inhomogeneous traveling waves can be obtained only if \( f(u) \) is in the form of a step which falls abruptly at some value of \( u = u_\infty \) by an amount \( \Delta \zeta \) in such a way that the relative width of the step \( \Delta u \) is less than its height \( \Delta u/\Delta u \ll \Delta \zeta/\zeta(0) \); no mechanisms ensuring such

\[\text{nonlinearity in optics (for } \Delta u/\Delta u \ll \Delta \zeta/\zeta(0)) \text{ are known at present.}\]

4. The situations changes in the case of oblique incidence of a wave near the total-internal-reflection angle or in the case of almost glancing incidence if the condition \( \zeta(0) - \zeta_0 \ll \zeta(0) \) is satisfied. Then, \( \zeta_{nt}(u) \) may pass through zero (and, consequently, \( F(u) \) may have a falling region) even in the case of small values of \( \Delta \zeta_0 \) and for ordinary types of nonlinearity. If \( f(u) = \epsilon(0) - \epsilon_0 u^2 \left( \epsilon_0 u^2 \ll \epsilon(0) \right) \), then \( \zeta_{nt}(u) = 0 \ll \epsilon_0 u^2 \) and if the condition \( \epsilon_0 u^2 \ll \epsilon(0) \) following from point 4 of the criterion is satisfied, Eq. (2) has solutions in the form of longitudinally inhomogeneous traveling waves:

\[
u_1 = \left[ u_1^2 + \frac{3\epsilon_0}{\epsilon(0)} \right]^{1/2}, \quad u_2 = \left[ u_2^2 - \frac{\epsilon_0}{\epsilon(0)} \right]^{1/2}, \quad B = \text{const} \quad \text{(the question of the stability of such solutions is still to be decided).}
\]

The solution \( u_1 \) in Eq. (8) corresponds to a wave which initially is apparently partly "pinned" to the interface and the solution \( u_2 \) behaves in the opposite manner. These waves can exist in the same range of parameters as hysteretic jumps\(^7\) in the case of falling nonlinearity (Ref. 7 gives also estimates of the necessary parameters); their existence should clearly result in a more complex general pattern of nonlinear reflection and refraction than in the case of a growing nonlinearity.

5. When a particle moves in a central attractive field with a potential proportional to \( -1/r \) truncated fairly abruptly (for example, exponentially) at a distance \( r_0 \), we find that in the limit \( t \to \infty \) we can have trajectories with \( r \to r_0 \) which throughout their existence approach the center or move away from it only once (this is an analog of longitudinally inhomogeneous traveling waves).

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