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CONTRIBUTION TO THE THEORY OF A PARAMETRIC GENERATOR OF
SUBHARMONICS UP TO THE n-TH ORDER. TRANSIENT PROCESSES

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The paper discusses the equation of motion of a system and investigates
the dynamic behavior of the phase of a subharmonic under steady-state conditions
of oscillations and during its build-up process. The operating conditions for
automodulation of oscillations (self-pulling frequency) are also discussed.

INTRODUCTION

Increasing attention is recently being paid to the subharmonic operating conditions of a
two-circuit parametric oscillator [1-6] (with high multiplicities of subharmonics, i.e., with
n > 2), which make it possible to use it as a frequency divider, multistable parametron, phase
detector, phase quantizer, random number generator, etc. In connection with this, the study of
transient processes in such systems (in the first place those of phase which carries informa-
tion in a phase trigger and which is essentially responsible for broadening of the line) is of
special interest; such a study gives the key to the investigation of the stability of steady states,
fluctuation processes, the behavior of phase when an external synchronous signal is acting, etc.

The purpose of this work is to investigate the transient processes and stability of steady
states in a parametric generator of subharmonics (n > 2) as well as of the operating conditions
for self-pulling of frequency.

1. EQUATION OF MOTION AND THE STEADY-STATE
OPERATING CONDITIONS OF SUBHARMONIC OSCILLATIONS

In the literature the term two-circuit parametric oscillator refers to nonlinear (from
the standpoint of the energy-storage capacitive parameter) systems with two degrees of freedom
(for example, of the type of the system shown in Fig. 1; the capacitor C(t), which depends on
the voltage u across it, is here the nonlinear element). The system is excited by a strong
signal (the "pumping" signal) of frequency ω_p which is close to the sum of the natural frequen-
cies of the system Ω_2 = Ω_1 + Ω_2, where Ω_1 and Ω_2 are the "normal" natural frequencies (with
the constant component of the nonlinear capacitor C(u) taken into account). When the amplitude
of the pumping voltage A_p across the capacitor C(u) is sufficiently high, oscillations are excited
in the system whose fundamental frequencies — ω_1 and ω_2 — are close to the corresponding
natural frequencies Ω_1 and Ω_2, and their sum is equal exactly to the pumping frequency:
ω_1 + ω_2 = ω_p [5-8, 1].

The equation of motion of the system shown in Fig. 1, in the case when the natural fre-
quencies Ω_1 and Ω_2 are sufficiently far apart (i.e., when |Ω_2 - Ω_1| ≫ h, where h_1, h_2 are
transmission bands at frequencies Ω_1 and Ω_2), has the form

\[
\left( \frac{d^2}{dt^2} + \delta_1 \frac{d}{dt} + \Omega_1^2 \right) \left( \frac{d^2}{dt^2} + \delta_2 \frac{d}{dt} + \Omega_2^2 \right) u =
\]

\[
= \frac{1}{c_0} \frac{d}{dt} \left( \frac{d^2}{dt^2} + \Omega_2^2 \right) \left[ \Omega(t) - \frac{d}{dt} q(t) \right].
\]

(1)

where \( \delta_1 = h_1/\Omega_1 = 1/Q_1 \) (1 = 1, 2) are the partial damping factors (with the nonlinear resistance
shunting the capacitor C(u) taken into account since for the latter a reverse-biased p-n junction
is ordinarily used); Q_i are the partial Q-factors; u is the voltage on the nonlinear capacitor.
\[ q(u) = C_\omega u + \sum_{k=1}^\infty q_k u^{k+1}; \] (2)

\[ C_3 = C_2 + \left[ \frac{C_{11} C_{22}}{(C_{11} + C_{22})^2} \right]; \quad \Omega_{11}^2 = \left[ \left( \frac{1}{L_1} \right) + \left( \frac{1}{L_2} \right) \right]/(C_{11} + C_{22}) \]

is the frequency of the "current" resonance; \( I(t) \) is the given pumping current (which in the absence of oscillations creates a voltage \( \omega_0(t) \) on the capacitor \( C(u) \) with the amplitude \( A_p \) and frequency \( \omega_p \).

In the steady state condition of parametric oscillation, oscillation establishes itself in the system which, in the first approximation, can be written in the form

\[ u(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) + \]
\[ + A_3 \cos(\omega_3 t + \phi_3). \]

The dependence of the amplitudes \( A_i \) of the parametrically excited combination oscillation, frequency \( \omega_i \) and phases \( \phi_i \) (\( i = 1, 2, 3 \)) on the pumping parameters and the parameters of the system itself have been investigated in [6-8, 1].

Let us introduce the so-called normalized notation. Let \( \omega_1/\omega_p = m_1, \omega_2/\omega_p = m_2 \) (\( m_1 + + m_2 = 1 \)) everywhere in the following all frequencies will be normalized with respect to the pumping frequency \( \omega_p \).

It has been shown in [1] that in the case when \( m_1 \), which in the general case is an irrational number, is close to one of the rational numbers \( l/n \), where \( l \) and \( n \) (not very large natural numbers) are not multiplies of each other, \( 0 < l < n \), and an auto-synchronization of oscillations can take place in the system as a result of which the combination oscillation (3) changes into a subharmonic oscillation of the \( n \)-th order:

\[ \ddot{u}(t) = A_1 \cos\left(\frac{\omega_p}{n} t + \tilde{\phi}_1\right) + A_2 \cos\left(\frac{\omega_p}{n} \frac{n-l}{n} t + \tilde{\phi}_2\right) + A_3 \cos(\omega_3 t + \tilde{\phi}_3). \] (4)

i.e., the frequencies are determined by the relations \( \tilde{m}_1 = l/n, \tilde{m}_2 = (n-l)/n \). This phenomenon is due (in the first approximation) to the interaction between the frequencies of generated oscillations on the one hand, and the beats of the \((n-1)\)-th order on the other hand (\( \omega_{(n-1)} = -(n-1)\omega_1 + + l\omega_1 \) is situated beside \( \omega_1 \), and \( \omega_{(n-1)} = (n-l)\omega_1 - (l-1)\omega_2 \) is situated beside \( \omega_2 \)). The existence of high-order combination frequencies is associated with the presence of nonlinearity (in the first approximation the \((n-2)\)-th term of expansion (2) is responsible for the generation of frequencies \( \omega_{(n-1)} \) and \( \omega_{(n-2)} \)).

In this case the shortened equations of motion can be written in the form

\[ \dot{A}_i \left( 2 \frac{\Delta \omega_i + \phi_i}{h_i} + \mu m_j \cos \gamma \right) + \frac{\sigma_i A_i A_p}{C_i \delta_i} \cos \varphi \right) = 0, \]

\[ 2 \frac{\dot{A}_i}{h_i} + \dot{A} \left[ 1 + \frac{\phi_i}{h_i} + (-1)^j \mu m_j \sin \gamma \right] + \frac{\sigma_i A_i A_p}{C_i \delta_i} \sin \varphi = 0, \]

where \( 3-j = 1, 2; \varphi = \varphi_1 + \varphi_2; \right \Delta \omega_i - \Delta \omega_i - \Omega_j; \gamma = \nu \varphi_1 - \nu \varphi \); \( \nu \) is the nonlinear part of the damping factor \( \delta \);

\[ \theta = \bar{m}_i - m_i = \frac{l}{n} - m_i = m_2 - \frac{n-l}{n}; \]

*Equations (5) are written under the assumption that the basic mechanism which limits the oscillations in the system is the dissipative mechanism, i.e., limitation of oscillations on account of increases of attenuation when the amplitude increases (see, for example [3]). Under this condition, in particular, we can neglect the reaction on the pumping and the detuning mechanism, i.e., \( A_3 = A_3 = A_p, \varphi_3 = 0 \). This condition is not of principal importance; the characteristic features of the process of auto-synchronization are maintained also for any other mechanism limiting the oscillations.
\( \theta \) is the normalized detuning between the "nonperturbed" frequencies of oscillation \( m_1 \) and \( m_2 \) and the corresponding frequencies of the steady-state subharmonic oscillation \( l/n \) and \( (n-l)/n \):

\[
\theta = \frac{n!}{2^{n-2} n! (n-l)!} \frac{\sigma_{n-2} A_1 A_2^{n-1}}{C_1 C_2 \delta_1 \delta_2},
\]

(7)

where \( C_1 \) and \( C_2 \) are the "normal" capacitances \( (C_1 = C_{11} + C_0 \text{ when } C_0 \ll C_1 + C_2) \).

Equation (7) differs from the corresponding equations, describing the "nonperturbed" mode of oscillation (3), by the presence of the "synchronizing" terms of the order \( \mu \) which contain phase \( \gamma = n \Phi_1 = \frac{1}{2} \Phi_{\Sigma} \) and whose appearance is due to the fact that the harmonic components with frequencies \( \omega_{11} \) and \( \omega_{12} \) are taken into account. It has been shown in [6-8, 1] that in the case of ordinary "nonperturbed" parametric generation of oscillation the phases \( \Phi_1 \) and \( \Phi_2 \) cannot be determined separately; only their sum \( \Phi_{\Sigma} \) can be found. However, the phases \( \Phi_1 \) and \( \Phi_2 \) can be determined separately for the subharmonic oscillation (4) on whose frequencies an additional condition \( \omega_1/\omega_2 = l/(n-l) \) is imposed.

In the general case the exact solution of the set (5) cannot be found. However, in the case when the parameter \( \mu \) is small

\[
\mu < 1,
\]

(8)

which for \( n > 3 \) practically always takes place*, auto-synchronization can be considered to be a small perturbation which causes a small variation of the amplitudes \( A_1 \) and \( A_2 \) and the phase \( \Phi_{\Sigma} \). Introducing the notation \( \Delta \xi = \xi - \xi \) (where \( \xi \) is any of the quantities \( A_1 \) and \( \Phi_{\Sigma} \) and assuming that \( \Delta \xi \ll \xi \), we can linearize Eq. (5) with respect to these small increments (the "secondary contraction"):

\[
\begin{aligned}
\frac{\partial \Phi_{\Sigma}}{\partial h_1} + 2Q \Delta \Phi \left( \frac{\Delta A_1}{A_1} - \frac{\Delta A_2}{A_2} \right) + \Delta \Phi_2 + \mu m \cos \gamma = & (-1)^i 2Q h_1 \frac{h_2}{h_1}, \\
\frac{\partial \Delta A_1}{\partial A_1} + \frac{\partial \Delta A_2}{\partial A_2} + \frac{\partial \Delta A_1}{A_1} \frac{\partial A_2}{A_2} \Delta A_1 + \Delta A_2 + \frac{x \Delta (A_1^2 + A_2^2)}{2 A_1^2 + A_2^2} - 2Q \Delta \Phi \Delta \Phi_2 &= (-1)^i \mu m \sin \gamma = 0,
\end{aligned}
\]

(9)

where \( h_{\Sigma} = h_1 + h_2; Q = \Omega_{\Sigma}/h_{\Sigma} \) is the generalized Q-factor; \( \Delta \Phi = (\omega_{\Phi} - \Omega_{\Sigma})/\Omega_{\Sigma} \) is the normalized detuning of the pumping frequency; \( x = 1 - A_0^2 \left( 1 + 4Q^2 \Delta \Phi^2 / \Delta \Phi^2 \right) \) is the parameter which characterizes the degree of regeneration of the system; \( A_0 \) is the threshold pumping amplitude; \( i = 3-j = 1, 2 \).

We shall be mainly interested in the behavior of the system close to the zero detuning of the pumping frequency (\( \Delta \Phi = 0 \)), i.e., in the center of the regeneration band where the amplitudes of oscillation \( A_1 \) are maximum and where, consequently, by virtue of Eq. (7) auto-synchronization is most effective. Moreover, in this region the solution has a most simple and descriptive form. For steady-state conditions of auto-synchronization (i.e., \( \Delta A_1 = \Phi_1 = 0 \)) we obtain

\[
\Phi_{\Sigma} = \Phi_2 + \frac{1}{n} \arccos \left( -\frac{\phi}{\theta_0} \right) + 2k \frac{\pi}{n},
\]

(10)

\[
\frac{\Delta (A_1^2 + A_2^2)}{A_1^2 + A_2^2} = \pm \frac{1}{x} \frac{2Q h_1^2}{h_1 h_2} \sqrt{\theta_0^2 - \theta_0^2},
\]

\[
\frac{\Delta A_1}{A_1} = \frac{\Delta A_2}{A_2} = \frac{1}{2} \frac{\phi}{\theta_0},
\]

\[
\Delta \Phi_2 = \frac{1}{2} \mu \frac{\phi}{\theta_0} + \theta Q \frac{h_1^2 - h_2^2}{h_1 h_2},
\]

(11)

where

\[
\theta_0 = \mu \frac{h_1 h_2}{2Q h_1^2} (m_2 - m_1),
\]

(12)

where \( k \) is an arbitrary integer.

*For \( n = 3 \mu = (3/2) (A_1 / A_0) \) (\( A_0 \) is the threshold pumping amplitude). Therefore, for \( n = 3 \) the condition (8) is satisfied only when \( A_1 \ll A_0 \).

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It is evident from Eq. (10) that the quantity \( \Theta \) is the half-width of the "internal" band of auto-synchronization (i.e., of the band achieved by retuning of the parameters of the system itself; the retuning is obtained by a variation of the magnitude of the internal detuning \( \Theta \)). The value of the "external" band of auto-synchronization, i.e., of the band obtained by retuning the pumping frequency \( \omega_p \) can be greater by several orders of magnitude than the internal band \( \Theta \).

As was shown in [1-3], the external band of auto-synchronization attains maximum value (up to the band of oscillation) under the condition \( Q_1 = Q_2 \), or \( h_1/h_2 = \lambda/(n-l) \). Outside the band \( \Theta \), i.e., for \( |\Theta| > \Theta \), auto-synchronization ceases abruptly. However, inside the band, as it follows from Eq. (10), for each value of \( \Theta \) there exist two steady-state modes; it is shown below that the upper signs in relations (10) correspond to a stable mode, and the lower signs correspond to an unstable mode. It must be noted that the quantity \( \Delta (A_1^2 + A_2^2)/(A_1^2 + A_2^2) \) corresponds approximately to the relative increment of output power \( \Delta W/W \). Therefore, an additional transfer of energy from the pumping frequency to the oscillation frequency and not conversely naturally corresponds to the stable mode.

It also follows from relations (10) that a possible stable states of the phase of a subharmonic oscillation separated by the same intervals \( 2\pi/n \) exist within the limits from 0 to \( 2\pi \).

2. THE MODE OF ESTABLISHMENT OF SUBHARMONIC OSCILLATIONS

In order to investigate the dynamic behavior of the processes in the system and the stability of steady states it would be worth-while to simplify further Eq. (9) which represent a set of the fourth order which is nonlinear with respect to the phase. It can be readily shown that in the case \( \Delta_p = 0 \) set (9) "splits" in such a manner that for the subharmonic phase \( \varphi_1 \) of greatest interest we can obtain the following second-order equation which does not contain any other unknown functions:

\[
2Q \frac{\ddot{\varphi}_1}{\omega_p^2} + \frac{\dot{\varphi}_1}{\omega_p} + \Theta \cos n\varphi_1 = -\Theta.
\] (13)

Under the additional condition

\[ n\Theta \ll 1/Q, \] (14)

which is analogous to the condition of "weak" external synchronization in an ordinary oscillator (when the band of synchronization is much smaller than the transmission band) and practically coincides with the fundamental condition (8), the behavior of the phase can be described by the following equation which is even simpler:

\[
\frac{\dot{\varphi}_1}{\omega_p} + \Theta \cos n\varphi_1 = -\Theta
\] (15)

[This is the third shortening of the original equation (1)]. It can be shown that under condition (14), Eq. (15) is valid also for any detuning \( \Delta_p \) with an accuracy of a shift of phase \( \varphi_1 \) which is of no importance to us.

The phase portrayal of Eq. (15) for different values of internal detuning \( \Theta \) is shown in Fig. 2. We can readily see that the stable singular points correspond to stationary values of the phase (10) with the upper sign and the unstable singular points correspond to stationary values of the phase (10) with the lower sign. Solution (10) is written in such a manner that the upper sign for the stationary amplitude increment \( \Delta (A_1^2 + A_2^2)/(A_1^2 + A_2^2) \) corresponds to the upper sign for the stationary phase \( \varphi_1 \), and the lower sign for the stationary amplitude increment \( \Delta (A_1^2 + A_2^2)/(A_1^2 + A_2^2) \) corresponds to the lower sign for the stationary phase \( \varphi_1 \). Hence, it follows that a positive amplitude increment is stable and a negative amplitude increment is unstable. By solving Eq. (15), linearized in the region of any stable singular point,

\[
\frac{\varphi}{\omega_p} + n\gamma \frac{\dot{\varphi}}{\omega^2} - \varphi^2 (\varphi - \varphi_{1,\text{sta}}) = 0,
\] (16)

and by obtaining the solution in the form \( \varphi_1 - \varphi_{1,\text{sta}} \sim \exp(-t/\tau\Theta) \), we obtain the characteristic

*In Eq. (13) the terms of the order of \( \mu^2 \) were neglected.
phase settling time

\[ \tau_p = \frac{1}{\nu \omega_p} \sqrt{\varphi^2 - \varphi^i}. \]  

Let us now discuss the following problem. In the derivation of the secondarily shortened Eqs. (9) from the initial equations (5) we have taken the steady-state mode of unperturbed generation of oscillation as the "zeroth" solution for which we have sought corrections. Therefore, Eqs. (13) and (15) describe the behavior of the phase of the subharmonic for already established regeneration. However, in some application of the generator of subharmonics, in particular when used as the n-stable parametron, we have to deal with the pulsed mode of operation when for each new cycle the generator has to be re-excited. In this case it is necessary to know the dynamic behavior of auto-synchronization when oscillations are excited in the system "from zero".

The initial shortened equations (5) are valid at any instant of time and for any amplitudes. The assumption that the generation of oscillation has established itself, used in the derivation of relations (9) from Eq. (5) is not an essential one; it permits only to assume that the coefficients in Eqs. (9) are constant and facilitates their determination. However, the essential assumption (from the standpoint of making the secondarily shortening possible) is only the one about the smallness of \( \mu \), i.e., about the smallness of the amplitudes of beats in comparison with the amplitudes of generated oscillation. However, if the condition (8) is satisfied during the settling period it is satisfied to an even greater extent under steady-state operating conditions since \( \mu \approx A_1^{n-2} \). Therefore, secondarily shortened equations of the type (9) can be obtained for any transient process in the system. In the general case they will represent equations with time-invariant coefficients which are determined by the amplitudes and phases of the unperturbed mode establishing the generation of oscillation assumed to be known functions of time. The solution of such equations in their complete form is not strictly essential; we can obtain the following simple equation for the parameter of greatest interest which is the phase \( \varphi_1 \)

\[ \frac{\varphi_1}{\omega_p} + \varphi_2 n^{-1} \cos n \varphi_1 = - \varphi_1, \]

where \( x(t) = A_{gen}(t)/A_{gen\, sta}, A_{gen} \) is any one of the amplitudes \( A_i \) (\( i = 1, 2 \)), \( A_{gen\, sta} \) is the established value of \( A_{gen} \).

Formally, Eq. (18) coincides with Eq. (15) if in the latter we introduce \( \varphi_{ef}(t) = \varphi_2 x(t)^{n-1} \) in place of the constant \( \varphi_1 \). The meaning of the quantity \( \varphi_{ef}(t) = \varphi_2 x(t)^{n-1} \) is evident if it is recalled that the parameter \( \mu \) appearing in \( \varphi_1 \) of Eq. (12) is proportional, according to Eq. (7), to the \( (n-2) \)-th power of the amplitude of generated oscillation. The shape of \( \varphi_{ef}(t) \) for different values of \( n \) is shown in Fig. 3b; for comparison, Fig. 3a shows the curve of the dependence of \( x(t) \). For \( n \gg 1 \), at a certain instant of time when the amplitudes of generated oscillation are very close to their steady-state values, the function \( \varphi_{ef}(t) \) suffers a rather sharp jump.

If the instant of the jump is defined as the point in which \( d^2 \varphi_{ef}/dt^2 = 0 \), and if we approximate the relative amplitude of generated oscillation \( x(t) \) near the established mode by the function \( x(t) = 1 - e^{-t/\tau_{gen}} \), where \( \tau_{gen} \) is the settling time of the amplitudes of generated oscillation, then we can readily obtain that the jump of \( \varphi_{ef}(t) \) (i.e., as if "switching-in" of the auto-synchronization mechanism) occurs at the time instant \( \tau_{sw_1} = \tau_{gen} \ln (n-2) \) (when the reference point is from the point where the approximated function vanishes). In the region of the switching-in instant \( \tau_{sw_1} \) and for sufficiently large values of \( n \) the function \( \varphi_{ef} \) itself has the form

\[ \varphi_{ef}(t) \approx \varphi_1 \exp \left[ - \exp \left( \frac{-t - \tau_{sw_1}}{\tau_{sw_1}} \right) \right]. \]

It is evident from Eq. (19) that up to the jump \( \varphi_{ef}(t) \) is very small (for example, for \( t = \tau_{sw_1} - 2\tau_{gen} \varphi_{ef} \approx \varphi_1 \times 10^{-3} \)). The duration of the jump (when \( \varphi_{ef} \) varies from \( e^{-c} \) to \( e^{-1/c} \)) is equal
Fig. 3. Curves showing the dependence of oscillation amplitude $x(t) = A_{gen}(t)/A_{gen\, sta}$ and the effective band of auto-synchronization $\vartheta_{\text{ef}}(t)$ on time.

Fig. 4. Amplitude-phase portrayals of a generator of the fourth-order subharmonic ($n = 4$)

a) for $\dot{\vartheta} = 0$; c) for $0 < |\dot{\vartheta}| < \vartheta_0$ and for a parametric generator in the absence of auto-synchronization ($\vartheta_0 = 0$); b) for $\dot{\vartheta} = 0$; d) for $\vartheta \neq 0$.

to $2\tau_{gen}$, i.e., it is much smaller than the settling time of the phase $\tau_\varphi$ as it follows from Eq. (14) if we take into account that $\tau_{gen}$ is of the same order, and usually even smaller, as the order of the time constant of the system without regeneration $2/\hbar_\varphi$. We can thus assume that for $n \gg 1$ auto-synchronization switches itself in by a jump and this takes place for already practically fully established generation of oscillation (this can be considered to be valid, to some approximation also for $n = 3$).

In the case when the initial amplitudes of generated oscillation differ greatly from their steady-state values it is convenient to illustrate the behavior of the system by means of the so-called amplitude-phase portrayal of the system which is determined by an equation of the type $dx/d\varphi = F(x, \varphi)$. If the behavior of the amplitudes of generated oscillation in time can be described by the equation $\dot{x}/\omega_p = x/f(x)$, where $f(0)$ is the increment of growth at $x = 0$ and $f(1) = 0$ ($-1/\omega_p) f''(x=1) = \tau_{gen}$, then from Eq. (18) the equation of the system in the amplitude-phase plane has the form

\[
\frac{dx}{d\varphi} = -x/f(x)/\left(\vartheta + \vartheta \varphi^{n-2} \cos n\varphi_0\right).
\]  

(20)

Figure 4a and b shows the amplitude-phase portrayals for different values of detuning $\dot{\vartheta}$ (for the particular case of fourth-order auto-synchronization, i.e., of division by 4) in polar coordinates — the representative point of the trajectory is the end point of the radius-vector of length $x$. 

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which makes the angle \( \phi_1 \) with some axis. For comparison, Fig. 4b and d shows corresponding portrayals for "unperturbed" generation of oscillation, i.e., in the case when there is no auto-synchronization \( \phi_0 = 0 \).

3. THE SELF-PULLING MODE OF FREQUENCY (AUTOMODULATION)

Outside the band of auto-synchronization, i.e., for \( |\phi| > \phi_0 \), the interaction between the fundamental frequencies of oscillation \( \omega_1 \) (\( i = 1, 2 \)) and the corresponding higher combination tones \( \omega_{ki} \) does not cease. It leads to modulation of the phase and amplitude of oscillation ("automodulation") and to a shift of the mean value of oscillation frequency ("self-pulling" of frequency) which is analogous to frequency pulling in an ordinary nonautonomous oscillator.

When the condition \( n\phi < 1/\Omega \) is satisfied, then, similar to Eq. (14), the behavior of the phase of oscillation in the self-pulling mode is described as before by Eq. (15) (Fig. 2c) whose integration for \( |\phi| > \phi_0 \) gives

\[
\varphi_t = \frac{2}{n} \arctan \left( \sqrt{\frac{\theta + \phi_0}{\theta - \phi_0}} \right) \left( \frac{1}{2} n\omega_p(\theta^2 - \phi_0^2) \right),
\]

\[
\dot{\varphi}_t = -\frac{\phi_0}{1 - (\theta^2/\phi_0^2)} \left[ (\theta^2/\phi_0^2) \cos(n\omega_p(\theta^2 - \phi_0^2)) \right]
\]

(with an accuracy not exceeding the inessential integration constant).

Therefore, the frequency of oscillation \( (m_1 = \theta^2/\phi_0^2) \) is a periodic time function with modulation frequency \( \omega_m \),

\[
\omega_m = n\omega_p(\theta^2 - \phi_0^2)
\]

(23)

(It is of interest to compare Eq. (23) with formula (17) which determines the settling time of the phase of the subharmonic). We can determine from (22) the time average value of frequency \( \bar{m}_t = \omega_t/\omega_p \):

\[
\bar{m}_t = \frac{\theta_0}{n} \left[ \frac{\theta_1}{\omega_p} = \frac{\theta_0}{n} - \gamma \theta^2 - \phi_0^2 \right]
\]

(24)

(see Fig. 5a); for \( |\phi| > \phi_0 \) the averaged frequency \( \bar{m}_t \) tends to its "unperturbed" value \( (\theta^2/\phi_0^2) \). Relation (24) is completely analogous to the corresponding relation for an ordinary non-autonomous oscillator.

As it was already stated above self-pulling of frequency is accompanied by modulation of both frequency (phase) and amplitude with modulation frequency \( \omega_m \) (23). As a result of this modulation together with the central oscillation frequency \( \omega_1 \) there are produced side components \( \omega_1 \pm \omega_m, \omega_1 \pm 2\omega_m \) and so on. Of course, the most intense are the nearest components \( \omega_1 \pm \omega_m \). It can be shown that for \( Q\Lambda_p \ll 1 \) phase modulation predominates. Considering this fact and using relation (21) we can readily represent the oscillation at the lower oscillation frequency in the approximate form:

\[
\cos \left( \frac{\theta}{n} \omega_1 \varphi_1(t) \right) \approx \cos \omega_t t + a_m [\cos(\omega_1 + \omega_m) t - \cos(\omega_1 - \omega_m) t],
\]

where

\[
a_m = \left( \frac{1}{n} \right) (\theta - \gamma \theta^2 - \phi_0^2) / \theta_0
\]

(25)

Figure 5b shows the behavior of the ratio of the amplitude of the first side frequency to the amplitude of the central frequency, \( a_m \), as a function of detuning \( \theta \). It follows from Eq. (25) that the maximum value of \( a_m \) is equal to \( 1/n \) when \( |\theta| = \phi_0 \); when \( |\theta| \) increases \( a_m \) decreases to zero. Therefore, if in the case of pulling the frequency of an ordinary oscillator by the frequency of an external signal we could consider in some sense the oscillations in the system as beats (since then at least two fundamental frequency components with approximately equal amplitudes can be present in the spectrum), then in our case, i.e., in the case of self-pulling of frequency, we are not dealing with beats but with modulation.

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Fig. 5. Dependence of the average frequency of oscillation \( \bar{\omega}_i \) and the relative amplitude of the side frequencies \( \Delta \omega \) on the internal detuning \( \delta \) in the self-pulling mode of oscillation frequencies (\(|\theta| > \theta_0\)).

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