PHASE FLUCTUATIONS IN A DOUBLE-LOOP PARAMETRIC
SUBHARMONIC OSCILLATOR

A. Ye. Kaplan

Fluctuation characteristics of a system working in steady state are investigated. These include smearing of the stationary phase, statistical displacement and widening of the frequency line of the subharmonic caused by strong, slow-varying noise, and phase jumps when the oscillator is working as an n-stable parametron.

1. The literature describes double-loop, parametric subharmonic oscillators of high order (n > 2) (see [1–5], listed in [1]) and lists their various applications, in particular as frequency dividers and n-stable parametrons. Because of these applications, a study of fluctuation processes in such systems is required. In particular, [2, 3] investigated the probability of establishment of a given phase of third order subharmonic as a function of pre-oscillatory conditions, when the system is triggered by a small synchronous signal which acts simultaneously with noise. However, the questions related to the effects of fluctuations on steady-state subharmonic generation have not been investigated, even today. Here we should consider, first of all, such parameters as the random frequency shift away from the exact value of subharmonic frequency (l/n)ω₀, the widening of the spectral line caused by fluctuations (both are important in frequency division), statistical characteristics of the phase "jitter" (when the oscillator is used as a phase transformer in precision measuring devices) and, finally, the average number of phase discontinuities per unit of time which determines the reliability of an n-stable parametron.

2. Since for our purposes the most important parameter is the phase of the oscillation, whose fluctuations are also mainly responsible for the displacement and the widening of the spectral line, we shall only be interested in phase fluctuations. It is possible to show that under the condition of "weak" autosynchronization (see [1]) the equations for amplitude and phase are "solutions" of each other. It is shown in [1] that the behavior of the phase Φ of the subharmonic may be described by an equation of the type:

\[ \frac{\Phi}{\omega_p} + A_0 \cos n\Phi + \delta = 0, \]  

where ω₀ is the pump frequency (input frequency in division), A₀ is the internal autosynchronization band and δ is the internal detuning.
When affected by noise, the phase equation becomes

\[ \frac{\phi}{\omega_p} + \theta \cos \theta \phi + \phi = \frac{\xi(t)}{\omega_p}, \tag{2} \]

where \( \xi(t) \) is assumed to be stationary gaussian noise with the spectrum

\[ S_\xi(v) = \frac{\omega_0}{1 + (v/\rho)^2} + 2D, \tag{3} \]

and \( \omega_0 \gg p, \omega_0 \gg D \) (Fig. 1a).

\[ F_1 \]

\[ F_2 \]

Fig. 1. Graphs of spectral density \( S_\xi(v) \) of the noise \( \xi \) which perturbs the system (a), and of spectral density \( S_\phi(v) \) of the phase of subharmonic \( \phi \) (b).

In principle, it is possible to compute the parameters of \( \xi(t) \) in terms of statistical parameters of real physical noise in the system. For our purposes, however, it is sufficient to point out that in the absence of autosynchronization (\( \theta_0 = 0 \)) the noise with the spectrum given by Eq. (2) causes the so-called spontaneous widening of the spectral frequency line of (\( \phi \)), \( \Delta \omega_{\text{spont}} = 2D \) (fast and small frequency drifts), and a technical widening of the line, \( \Delta \omega_{\text{techn}} = \gamma (\pi/2) \omega_0 \rho \) (slow and large frequency drifts), where \( \Delta \omega_{\text{techn}} \gg \Delta \omega_{\text{spont}} \). It may then be assumed that these quantities are known for the initial, "unperturbed" oscillation (i.e., in the absence of autosynchronization).

The technical widening of the line is caused mainly by fluctuations of the parameters of the system and of the pump. Strictly speaking, these fluctuations are not additive noise, but when the initial equations are linearized [1] and when their value is assumed to be small in comparison to the parameters themselves, they can be represented as the additive power \( \xi(t) \) in Eq. (2).

3. First we shall consider the linear case, i.e., the "jitter" of the phase around a stable, stationary value \( \phi_0 = -(1/n) \arccos (-\theta / \theta_0) + 2k (\pi/n) \) in the presence of low noise, \( \Delta \omega_{\text{techn}, \text{spont}} / \omega_p \ll \theta_0 - |\theta| \). In this case, linearizing Eq. (1) in the vicinity of any stable state, we obtain

\[ \phi + (\phi / \tau_p) = \xi(t), \tag{4} \]

where \( \tau_p = 1/n \omega_p \sqrt{\theta_0^2 - \theta^2} \) is the time required to establish the subharmonic phase [1].

From this we can obtain the spectrum of fluctuation of phase \( \phi \rightarrow S_\phi(v) \) (Fig. 1b):

\[ S_\phi(v) = \frac{S_\xi(v)}{v^2 + (1/\tau_p^2)} \tag{5} \]
The mean squared phase deviation, or variance $\sigma_\varphi^2 = \overline{\Delta \varphi^2}$, is defined by the formula

$$\sigma_\varphi^2 = \int_0^\infty S_\varphi (v) \, dv.$$  

If only spontaneous widening of the line of initial oscillation exists, we obtain

$$\sigma_{\text{spont}} = \sqrt{\pi \tau_\varphi} = \sqrt{(\pi/2) \Delta \omega_{\text{spont}} \tau_\varphi}.$$  

(6)

If, however, the fluctuations causes both the spontaneous and the technical widening, then

$$\sigma_{\text{tech}} = \Delta \omega_{\text{tech}} \tau_\varphi.$$  

(6')

Thus, with autosynchronization, the "smearing" of the phase $\varphi$ resembles in some respect the widening of a frequency line encountered in normal oscillation. We have small (6) but "fast" (rate $\sim 1/\tau_\varphi$) phase deviations ("spontaneous" phase widening), and strong (6') but "slow" (rate $\sim \rho$) deviations ("technical" widening of the phase).

When the boundary of autosynchronization is approached ($|\theta| \rightarrow \theta_0, \tau_\varphi \rightarrow \infty$), the smearing of the phase increases, and when the variance $\sigma_\varphi \sim 1$, our assumptions become invalid because then it becomes necessary to take into account the nonlinearity of Eq. (1).

4. Even when the noise is small, the nonlinearity must be taken into account in order to compute the phase discontinuities and the resulting drift of the average frequency of oscillation and the widening of its spectral line. If we note that the phase discontinuities are caused mainly by the "technical" fluctuations which are most important from the practical point of view, then the problem is significantly simplified because of the slow rate of technical fluctuations, allowing us to use the quasi-static approach. This enables us to compute the instantaneous frequency at any moment of time and, on the other hand, to evaluate the amount of the frequency shift as an average of the frequency drifts, and the line widening as the variance of these drifts.

*It should be noted that for investigation of phase discontinuities we could use the method given, for example, in [5], where, in particular, the effects of $\delta$-correlated noise on an oscillator which is synchronized by internal sinusoidal power are discussed.

Fig. 2. Curves of the variation of the probability density $w_0 (x)$ of "slow" frequency noise $\xi$ and of the average detuning of the frequency of oscillation $\Delta (x) = \overline{\Delta \omega} - (1/n)$ as functions of instantaneous detuning $x = \phi - \xi (t)$.

It is shown in [1] that the relative frequency drift $\Delta = \phi (t)/\omega_p = (\omega_1 (t)/\omega_p) - (1/n)$, away from the relative subharmonic frequency $1/n$, is given by

$$\Delta (x) = \begin{cases} 
-\sqrt{2x^2 - \delta_0^2}, & x > \delta_0, \\
0, & |x| < |\delta_0|, \\
+\sqrt{2x^2 - \delta_0^2}, & x < -\delta_0.
\end{cases}$$  

(7)

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where $x$ is the detuning (Fig. 2). In our case $x = \Theta - (\xi/t)/\omega_p$.

The average displacement of subharmonic frequency, $\langle \Delta \rangle$, is formed because the frequency drifts more often toward the nearest autosynchronization boundary. Taking into consideration Eq. (7), we can write

$$\langle \Delta \rangle = \int_{-\Theta_0}^{\Theta_0} \omega_{t}(x) \sqrt{x^2 - \Theta_0^2} \, dx - \int_{-\Theta_0}^{\Theta_0} \omega_{t}(x) \sqrt{x^2 - \Theta_0^2} \, dx,$$

(8)

where $w_{t}(x)$ is the probability density function of the noise $\xi$. In the case of gaussian noise

$$w_{t}(x) = \frac{1}{\sigma_{t} \sqrt{2\pi}} \exp \left[ - \frac{(x - \Theta)^2}{2\sigma_{t}^2} \right],$$

(9)

where $\sigma_{T} = \Delta \omega_{techn} / \omega_p$ is the relative technical width of the line of initial (unperturbed) oscillation (Fig. 2). The curves of $\langle \Delta \rangle / \Theta_0$ as functions of $|\Theta| / \Theta_0$ are shown in Fig. 3a for various noise levels, $\sigma_{T} / \Theta_0$.

![Fig. 3. Average value of random subharmonic frequency shift, $\langle \Delta \rangle$, (a) and widening of the spectral line, $\sigma_{subh}$ (b) as functions of internal detuning $\Theta$ for various noise levels $\sigma_{T} / \Theta_0$.](image)

Phase discontinuities which are accompanied by frequency drifts on both sides of the average frequency shift (8) cause a widening of the spectral line. In view of our previous statements, the relative width, $\sigma_{subh} = \Delta \omega_{subh} / \omega_p$ of the spectral line of a subharmonic can be written as

$$\sigma_{subh} = \langle \Delta^2 \rangle - \langle \Delta \rangle^2,$$

(10)

where

$$\langle \Delta^2 \rangle = \int_{-\Theta_0}^{\Theta_0} \omega_{t}(x) (x^2 - \Theta_0^2) \, dx + \int_{-\Theta_0}^{\Theta_0} \omega_{t}(x) (x^2 - \Theta_0^2) \, dx.$$

(11)

Curves of $\sigma_{subh} / \sigma_{T}$ as a function of $|\Theta| / \Theta_0$ are shown in Fig. 3b for various levels of noise, $\sigma_{T} / \Theta_0$.

In a general case the integrals in expressions (8) and (11) are not evaluated. The table shows a summary of approximate formulas for the most important practical case of low noise, $\sigma_{T} / \Theta_0 \ll 1$, when it is possible to assume that the autosynchronization is generally realized.
It follows from expressions (8)–(12) that for \( \theta_0 \geq \sigma_T \) the autosynchronization is practically immersed in noise. Since \( \theta_0 \) is related to the nonlinearity of the system and to the order of the subharmonic \( n \) [1], it is possible on this basis to compute the maximum orders of subharmonic oscillations which may result in the presence of real noise. Such estimates give \( n \sim 150-200 \).

\[
\begin{array}{|c|c|c|}
\hline
 & \theta = 0 & 0 \leq |\theta| < \theta_0 = \sigma_T \\
\hline
\langle \Delta \rangle & 0 & \left( \frac{\sigma_T}{\theta_0} \right) \frac{\theta}{\theta_0} - \left( \frac{\sigma_T}{\theta_0} \right) \frac{1}{\theta_0} \\
\hline
\sigma_{\text{subh}} & \frac{4}{\sqrt{2\pi}} \left( \frac{\sigma_T}{\theta_0} \right)^{1/2} e^{-\frac{a_T}{4\sigma_T}} & \left( \frac{8}{\sqrt{2\pi}} \left( \frac{\sigma_T}{\theta_0} \right)^{1/2} e^{-\frac{a_T}{4\sigma_T}} \right) \\
\hline
\end{array}
\]

(12)

5. When a subharmonic oscillator is used as a parametron, we are no longer interested in the frequency shift or in the widening of the line, but in the number of discontinuities of the phase \( \phi \) per unit of time, regardless of how much the phase drifts during the time that the autosynchronization is disrupted. Here we are dealing (in quasi-static case) with the problem of reaching the boundary \( \phi \pm \theta_0 \) by the random function \( \xi(t)/\omega_p \). For the spectrum of technical fluctuations of the type \( S_\nu = w_0/(1 + (\nu/\omega_p)^2) \), the required result is given by the formula [4]

\[
N = \frac{\nu}{2\pi} \left\{ \exp \left[ - \frac{(\theta_0 - \theta)^2}{2\sigma_T^2} \right] + \exp \left[ - \frac{(\theta_0 + \theta)^2}{2\sigma_T^2} \right] \right\},
\]

(13)

where \( \nu \) is the "average" frequency of fluctuations.

For \( \theta = 0 \),

\[
N = \frac{\nu}{\pi \sigma_T^2}. 
\]

(14)

6. Let us now give some estimates of the characteristics of a parametric subharmonic oscillator which are related to phase fluctuations. As an example, we shall consider a subharmonic oscillator whose initial oscillation has a relative technical line width \( \sigma_T \sim 10^{-3} \) (highly exaggerated) and for which the ratio of the internal autosynchronization band to the technical line width is \( \theta_0^2/\sigma_T \sim 10 \). This ratio is quite modest and can be easily realized in practice for up to 10–12th subharmonic.

When the tuning is optimum (\( \theta = 0 \)), we obtain from Eq. (6') that the phase smearing is \( \sigma_{\text{techn}} \sim 10^{-2} \).

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It follows from Eq. (12) that the average subharmonic frequency shift, is equal to zero and
that the widening of the subharmonic spectral line is
\[
\sigma_{\text{subh}} = \frac{\Delta \omega_{\text{subh}}}{\omega_p} = \sigma_f \frac{4}{(2\pi)^N} \frac{\sigma_f}{\langle \sigma_f \rangle} e^{-\sigma_f^2/\langle \sigma_f \rangle^2} \sim 10^{-11}. \tag{14a}
\]

If we assume that \( p/2\pi \) is of the order of 50 Hz, the average number of phase discontinuities
per second is, according to (14), \( N \sim 10^{-20} \) Hz, i.e., the average time between discontinuities
is \( 1/N \sim 10^{13} \) years.

7. In this article we have not considered the mechanism of transformation of fluctuations
of the pump waveform into the fluctuations of the subharmonic phase. A thorough investigation
of this problem is of extreme interest in itself, but here we shall limit ourselves to the follow-
ing few remarks. The amplitude fluctuations of the pump were taken into account indirectly in
all derivations because we have always assumed as given the width of the line of the initial ("non-
autosynchronized") oscillation which is, in particular, determined by these fluctuations. As far
as the fluctuations of the pump frequency are concerned, we can assume that the subharmonic
frequency follows accurately their strong and slow component (\( \tau < \tau_f \)), especially when it is
noted that in all cases of practical interest the width of the frequency division band is many
orders of magnitude larger than the line width of the pump. Therefore, the slow drifts of the
pump frequency (if they do not exceed the boundaries of the division band) do not contribute to
the widening of the line of relative subharmonic frequency, \( \omega_{\text{subh}}/\omega_p = l/n \), but the absolute
frequency \( \omega_{\text{subh}} \) follows these drifts with the coefficient \( l/n \). For this reason the technical
line width, as used in this article, has the meaning of the "characteristic" technical line width
of the oscillator itself which is due to the fluctuation of the oscillator parameters and of the
pump amplitude, but not due to the fluctuations of the pump frequency. It should be pointed out
that in a general investigation of subharmonic frequency fluctuations it is necessary to dis-
tinguish between the "characteristic" fluctuations and fluctuations introduced by the pump fre-
quency, whose accurate reproduction is the necessary property of the system. In fact, an ideal
frequency divider should follow accurately all changes of input frequency, and for this reason
the problem of fluctuating relative frequency \( \omega_{\text{subh}}/\omega_p \) is more important than that of \( \omega_{\text{subh}} \)
itself.

In conclusion the author wishes to express his gratitude to M. Ye. Zhabotinsky for his
attention to this work and for criticism of the results, and to S. A. Akhmanov for his valuable
suggestions.

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Submitted April 24, 1965