NONLINEAR MAGNETO-OPTICAL EFFECT IN VACUUM: 
INHOMOGENEITY-ORIGINATED SECOND-HARMONIC GENERATION 
IN DC MAGNETIC FIELD

Y. J. DING and A. E. KAPLAN

Department of Electrical and Computer Engineering, The Johns Hopkins University,
Baltimore, MD 21218, USA

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The photon-photon scattering predicted by quantum electrodynamics can give rise to 
second-harmonic generation of intense laser radiation in a dc magnetic field due to broken 
symmetry of interaction even in the "box" diagram approximation. This effect is possible only 
when the field system (i.e. optical wave + dc field) is inhomogeneous, in particular when a 
Gaussian laser beam (i.e. nonplane wave) propagates in either homogeneous or inhomoge-
neous dc magnetic field.

1. Introduction

Photon-photon scattering in a vacuum\textsuperscript{1,2} is perhaps one of the most fundamental 
quantum electrodynamics (QED) mechanisms which can give rise to nonlinear 
optical effects. From the classical point of view, the expected nonlinear interaction 
in the lowest approximation (the so-called fourth-order or "box" diagram, see 
Fig. 1) [see Eqs. (5) and (6)], essentially corresponds to the third-order non-
linearity.\textsuperscript{3} This interaction may contribute to the birefringence of the refractive 
index seen by a probe field under the action of either a dc magnetic (or electric) 
field\textsuperscript{4} or intense laser pumping,\textsuperscript{5} multiwave mixing,\textsuperscript{6} and merging of two photons 
into one (i.e. sum frequency generation)\textsuperscript{7} under the action of a dc electric field. It 
was also proposed that using sixth-order diagram, high-order harmonics\textsuperscript{8} and 
second-order subharmonic (photon splitting)\textsuperscript{4,9} can be generated under a dc 
magnetic field. To the best of our knowledge, no experimental work on this 
phenomenon has been done. This is basically due to the fact that the optical fields 
required to observe any of these effects, are enormously high and not available at 
the current level of laser technology.

From any realistic point of view, the only hope to achieve observable effects in 
photon-photon scattering is to use lowest-order nonlinear optical processes (i.e. 
those based on the box diagram). It has also become apparent quite a while ago that 
the presence of a dc field (either electric or magnetic) may greatly assist the 
interaction, first of all by having the dc field with the amplitude even higher than 
that of optical field, and by reducing the order of nonlinear interaction for optical
field (to the second-order process). This was perhaps one of the reasons why these processes have attracted so much attention.\textsuperscript{4-9} However, probably the most interesting dc field-assisted processes, such as the merging of the photons into one (e.g. the second-harmonic generation, SHG) and photon "splitting" in the presence of a dc field have been controversial subjects for a few decades. It has been shown first\textsuperscript{4,9} that the probability of the photon splitting vanishes in the box approximation. Almost immediately after that, the authors of Ref. 10 (see also earlier work\textsuperscript{11}) showed that the box diagram results are not vanishing; they predicted the probability of photon splitting to be about eleven orders of magnitude larger than that of Ref. 9. Since it is generally believed that the splitting and merging of the photons in the QED terms are described by the same formulas, vanishing results for splitting\textsuperscript{9} should also implicate vanishing probability for merging photons in a dc field as it has, for example, been obtained in Ref. 7; however, result\textsuperscript{12} indicated that the probability is not vanishing. Recently, the previous works on the subject were reviewed in Ref. 13, and a conclusion was drawn that all of them were wrong (although the results\textsuperscript{13} are much closer to those of Refs. 4 and 9). The new result from Ref. 13 is that (due to dispersion) the box diagram does not vanish, but the corresponding amplitude is much smaller than that from next order, hexagonal, contribution.

Most recently, we proposed\textsuperscript{14} the second-harmonic generation (SHG) of laser radiation in vacuum in the presence of a dc magnetic field. This result, however, has been disputed from both QED diagram\textsuperscript{15} and phenomenological\textsuperscript{16} points of view. After careful consideration\textsuperscript{17} of those arguments,\textsuperscript{15,16} we agreed with them to the extent that the result for "box" diagram vanishes only for collinear photons\textsuperscript{4,9} and when dispersion of vacuum is neglected. The main reason for the box diagram vanishing in a dc field is nonconservation of four-momentum in such an interaction.\textsuperscript{4,9,13} However, when the weak dispersion in vacuum (which always exists intrinsically) is taken into account,\textsuperscript{13} the condition of four-momentum conservation
appears to be less restrictive for second-order optical processes. This results in the fact that even in the box approximation, photon splitting should not completely vanish (see Appendix in Ref. 13 which points out that the dispersion can remove the identical vanishing of the effect although its amplitude remains smaller than that due to hexagonal diagram).

Thus, the initial expectations that the presence of a dc field per se would break the symmetry of interaction, and therefore allow for the significant lowest-order nonlinear interaction of EM wave to occur (which seems to be a driving stimulus behind all the works\textsuperscript{4,7,9–16}), proved to be wrong. However, a close consideration of works\textsuperscript{4,7,9–16} shows that all of them have dealt only with plane wave (or cophasing photons) propagation in a dc field. Can the inhomogeneity of dc field (as suggested in Ref. 17) or optical wave (i.e. nonplanar wave, or noncollinear photons) make a difference? In this paper, we show that the answer to this question is positive. It was first suggested by us\textsuperscript{17} that the nonvanishing effect can exist in spatially inhomogeneous dc field. In this paper, based on the equations for arbitrary field configuration (Sec. 2), we obtain general results for a Gaussian fundamental beam in inhomogeneous magnetic field with an arbitrary spatial distribution (Sec. 3). We show that nonvanishing SHG in the lowest (i.e. box) approximation can result from nonplanar wave (in particular, a Gaussian laser beam) propagating either in a homogeneous dc field (Sec. 3.1), or in inhomogeneous dc field (Sec. 3.2). In all the cases considered here, the driving nonlinear term is directly related to the inhomogeneity of the field system. The total power of SHG (or number of SHG photons) in such a system is \( 10^{16–10^{24} \text{ orders of magnitude higher than that due to hexagonal diagram contribution (Sec. 4), such that SHG appears to be truly of the “box diagram” nature. Although the interpretation of this “inhomogeneity”-originated nonlinearity in terms of the box diagram is not fully understood at the present stage, it is conceivable that the dispersion of vacuum which contributes to a slightly nonvanishing box diagram results even in the case of plane wave + homogeneous dc field in Ref. 13, can essentially be controlled and greatly enhanced by the field inhomogeneity (see a brief discussion in front of Sec. 3.1).

The basic motivation for investigating SHG is that since only a second-order effect for the optical field is involved, the laser power required to observe SHG is expected to be lower than in third-order effects for the optical field. The SHG is also to be measured at a frequency different from the fundamental frequency injected into the system, which may result in higher sensitivity. Our tentative analysis of experimental feasibility of the effect and masking effects is presented in Sec. 5.

2. Nonlinear Wave Equation

The Heisenberg-Euler Lagrangian (HEL)\textsuperscript{1,2} for photon-photon scattering in quantum electrodynamics (QED) theory, in the case of weak fields, can be expanded as
\[ L = L_0 + L_4 + L_6 + \cdots \]  

(1)

where \( L_0 = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) \) is the linear component of the HEL, \( L_4 \) is the first nonzero nonlinear component of the HEL, which in QED theory corresponds to the contribution from box Feynman diagram (Fig. 1), and can be expressed as

\[ L_4 = \frac{1}{2} \xi (\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E} \cdot \mathbf{B})^2, \]  

(2)

where \( \xi = \alpha/45\pi B_{cr}^2 = 2.6 \times 10^{-32} \text{ Gauss}^{-2} \) is a nonlinear interaction constant in the vacuum with \( \alpha = e^2/\hbar c = 1/137 \) the fine structure constant and \( B_{cr} = m_e^2 c^3/\hbar \) = 4.4 \times 10^{13} \text{ Gauss} \) the QED critical field. \( L_6 \) in Eq. (1) corresponds to the correction from hexagonal Feynman diagram, etc. The contribution to the nonlinearity (in particular to SHG) from this term \( L_6 \) (i.e. that of the fifth order in field amplitudes\(^{1,2} \)) is smaller than that originated from Eqs. (5) and (6) by many orders of magnitude for typical parameters (first of, the field intensities) considered in this paper, see Sec. 4.

Therefore, in this paper we only consider the lowest nonzero nonlinear term, \( L_4 \). Forming the action \( S = \int (L_0 + L_4) \, d^4x \), and taking the variation with respect to the four vector potential, one obtains the macroscopic equations which in the classical limit have the same form as classical Maxwell’s equations\(^{1,2,7,18} \):

\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c} \partial \mathbf{B}/\partial t = 0, \]  

(3)

\[ \nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{H} - \frac{1}{c} \partial \mathbf{D}/\partial t = 0. \]  

(4)

From the same procedure, one derives the following expressions for nonlinear components of the electric displacement \( \mathbf{D} \) and magnetic field \( \mathbf{H} \) as functions of electric field \( \mathbf{E} \) and magnetic induction \( \mathbf{B} \), i.e. nonlinear constitutive relations (NCR) (see e.g. Refs. 1, 2 and 18):

\[ \mathbf{D} = \mathbf{E} + \mathbf{D}^{NL}, \quad \mathbf{D}^{NL} = \partial L_4/\partial \mathbf{E} = \xi [2(\mathbf{E}^2 - \mathbf{B}^2) \mathbf{E} + 7 \mathbf{B} (\mathbf{E} \cdot \mathbf{B})], \]  

(5)

\[ \mathbf{H} = \mathbf{B} + \mathbf{H}^{NL}, \quad \mathbf{H}^{NL} = - \partial L_4/\partial \mathbf{B} = \xi [2(\mathbf{E}^2 - \mathbf{B}^2) \mathbf{B} - 7 \mathbf{E} (\mathbf{E} \cdot \mathbf{B})]. \]  

(6)

These equations are valid only if the nonlinear corrections \( \mathbf{D}^{NL} \) and \( \mathbf{H}^{NL} \) are small, which holds if \( |\mathbf{E}|, |\mathbf{B}| \ll B_{cr} \). It is important to note that the HEL in Eq. (1) is valid only for “slowly” varying fields (including constant fields) i.e. the fields with sufficiently low frequency compared to the critical QED frequency (see Ref. 19), which still makes it applicable for optical and even X-ray domains. In classical terms, the “slowness” condition is satisfied if\(^{18} \)
(h/mc) |∇ F| \ll |F|; \quad (h/mc^2) |∂F/∂t| \ll |F|, \quad (7)

where F represents electric or magnetic field. This immediately results in the condition \( \lambda \gg 2 \pi \lambda_c \), where \( \lambda_c = h/mc = 4 \times 10^{-3} \ \text{Å} \) is the Compton wavelength of electron.

It is obvious that a single monochromatic plane wave of infinite extent does not exhibit any nonlinear effects, since due to its properties, \( E^2 = B^2 \), \( E \cdot B = 0 \), the nonlinear terms in Eqs. (5) and (6) vanish. This “degeneracy” of the third-order nonlinearity is broken if either (i) the waves are noncollinear or/and multifrequency and/or (ii) a strong static field (e.g. a dc magnetic field) is present. Both cases can result in birefringence of the refractive index for a probe field. We show here that nonplanar (in particular, Gaussian) optical wave in combination with either homogeneous or inhomogeneous dc (e.g. magnetic) field can also give rise to second-order nonlinear optical effects similar to those found in dc field-biased nonlinear third-order materials.\(^3\) In general, the optical second-order nonlinearity can give rise to the generation of a third wave (at frequency \( \omega_3 = \omega_a \pm \omega_b \)) from two intense laser beams at frequencies \( \omega_a \) and \( \omega_b \) (i.e. the sum and difference frequency generation). Here, we consider only SHG in which \( \omega_a = \omega_b, \omega_3 = 2 \omega_1 \). However, our calculations can easily be generalized for the case \( \omega_a \neq \omega_b \).

Assuming that an unperturbed fundamental wave \( E_1 \) at the frequency \( \omega_1 \) propagates in vacuum in the presence of a dc magnetic field \( B_0 \), the nonlinear components in Eqs. (5) and (6) can be rewritten as

\[
D^{NL} = \xi [-2B_0^2 E_1 + 7B_0 (E_1 \cdot B_0)] + D^{(2)}, \quad (8)
\]

\[
H^{NL} = -2 \xi B_0 B_0 - 2 \xi [B_0^2 B_1 + 2B_0 (B_0 \cdot B_1)] + H^{(2)}, \quad (9)
\]

where the “second-order-optical” terms are

\[
D^{(2)} = \xi [-4E_1 (B_1 \cdot B_0) + 7B_1 (E_1 \cdot B_0)] , \quad (10)
\]

\[
H^{(2)} = \xi [-4B_1 (B_1 \cdot B_0) - 7E_1 (E_1 \cdot B_0)] . \quad (11)
\]

Here, we neglect the third-order nonlinearity since it cannot generate second-order optical effects (e.g. SHG). In principle, since the third-order effect may result in nonlinear refractive index, the possible phase mismatch between fundamental wave and SH may be expected, since the refractive index for a probe wave in vacuum can be a function of frequency and polarization of the wave.\(^4\,5\) The phase-matching condition can then be expressed as \( \triangle k = |2k_1 - k_2| = 2(\omega_1/c)|n(\omega_1) - n(2\omega_1)| \ll \pi/2L \) (\( k_2 \) is the wave vector of the second harmonic, \( L \) is the interaction length, \( n \) is the linear refractive index). Our estimate based on Refs. 4 and 5 show that for the laboratory field intensities available now and in the foreseeable future, this phase mismatch for SHG can always be neglected.
The terms in square brackets in Eqs. (8) and (9) are linear in optical field strengths. They result in vacuum birefringence of the refractive index for a weak probe field \(|\mathbf{E}_1| \ll |\mathbf{E}_0| \ll B_0\) under the action of a dc magnetic field.\(^4\) The first term on the right-hand side of Eq. (9) corresponds to dc corrections to the dc magnetic field induced by the magnetic field itself. Therefore, only the terms \(D^{(2)}\) and \(H^{(2)}\), Eqs. (10) and (11), give rise to the optical second-order nonlinearity. The wave equations for second-harmonic wave can now be obtained from Eqs. (3) and (4) by separating the field components at the SH frequency as

\[
\nabla \times \mathbf{E}_2 + \frac{1}{c} \frac{\partial \mathbf{B}_2}{\partial t} = 0,
\]

\[
\nabla \times \mathbf{B}_2 - \frac{1}{c} \frac{\partial \mathbf{E}_2}{\partial t} = F \cdot \exp[2i(k_1y - \omega_1t)]
\]

where the "driver"

\[
F \equiv \left[ \left( \frac{1}{c} \frac{\partial D^{(2)}}{\partial t} - \nabla \times H^{(2)} \right) \cdot \exp(2i(\omega_1t - k_1y)) \right]
\]

is nonlinear driving term which determines the entire second-order process; the exponential term in Eq. (13) is chosen in such a way that for a cw fundamental wave, \(F\) is time independent.

Because of the spatial anisotropy imposed by the magnetic field, SHG depends upon the propagation direction and the polarization of the fundamental optical wave with respect to \(B_0\). If the fundamental wave propagates along the direction of the dc magnetic field \(B_0\), then \(D^{(2)} = 0\), \(H^{(2)} = 0\), and the nonlinear effects are suppressed. The strongest interaction occurs when the laser radiation propagates in a direction normal to the dc magnetic field (see Sec. 3).

Now, let us show now that plane wave + a homogeneous magnetic field are not sufficient to produce SHG, i.e. the interaction vanishes in such a case. Consider a plane fundamental wave with an arbitrary polarization propagating along the \(y\)-axis such that

\[
\mathbf{E}_1 = \mathbf{E}_0 \exp[i(k_1y - \omega_1t)]; \quad \mathbf{B}_1 = (\mathbf{k}_1/|\mathbf{k}_1|) \times \mathbf{E}_1 ,
\]

where \(\mathbf{E}_0\) = constant is the vectorial amplitude of fundamental wave with its direction normal to the propagation direction \(y\), and \(\mathbf{k}_1\) is the wave vector of the field. An arbitrary homogeneous dc magnetic field can be represented as \(\mathbf{B}_0 = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}\). We can immediately see from Eqs. (10) and (11) that the component \(B_y \hat{y}\) does not contribute to the second-order nonlinearity since both \(\mathbf{E}_1\) and \(\mathbf{B}_1\) are normal to the axis \(y\). Substituting \(\mathbf{B}_0\), \(\mathbf{E}_1\) and \(\mathbf{B}_1\) expressed above into Eqs. (10) and (11), one can calculate the second-order nonlinearity \(D^{(2)}\) and \(H^{(2)}\). Substituting them into Eq. (13), one obtains

\[
F = 0.
\]

Therefore, with driving term in Eq. (12) vanishing, SHG vanishes too.
3. SHG in a Gaussian Fundamental Wave Propagating in dc Magnetic Field

Since a plane wave cannot produce SHG in a homogeneous dc magnetic field, the only factor left for the investigation now is inhomogeneity of the fields involved. Our calculations show that inhomogeneity, indeed, is a key factor for SHG in a dc field. In this section, we shall obtain equations for SHG originated from an inhomogeneous (specifically, Gaussian) fundamental wave propagating in a dc magnetic field with an arbitrary spatial distribution. We presume, however, that the dc field changes insignificantly in the cross-section of the beam in the area of strong interaction, which could be easily satisfied within paraxial approximation for the beam propagation adopted by us. For the sake of simplicity, we shall also use two-dimensional (2-D) dc field which does not depend on, e.g., the x axis.

As discussed above, the strongest effects are expected only when the laser beam propagates normally, or almost normally, to the dc magnetic field; therefore we always choose configuration whereby a component of dc magnetic field along the axis of propagation (i.e. the y-axis), the so-called longitudinal component, is zero in the center of the laser beam. In general case, the longitudinal component \( B_{y} \) of the dc field within the cross-section of the laser beam is always smaller than the transverse component \( B_{\perp} \) (=\( B_{z} \)). By virtue of the relatively small diffraction, the same is true for the longitudinal components of the electric field and magnetic induction of the laser field:

\[
\frac{(E_{1})_{y}}{E_{1}} \sim \frac{(B_{1})_{y}}{B_{1}} = O(\phi_{d}) \ll 1 ,
\]

where \( \phi_{d} \equiv \lambda_{1}/\pi v_{0} \) is the diffraction angle of the beam; \( \phi_{d} \ll 1 \) for paraxial approximation which is valid for all the cases of interest. Since the amplitude of driving terms (Eqs. (10) and (11)) is determined by the field scalar products, the contribution from the longitudinal terms, \((E_{1})_{y} \) \( B_{y} \) and \((B_{1})_{y} \) \( B_{y} \), is much smaller than that of the transverse terms, \((E_{1})_{\perp} \) \( B_{z} \) and \((B_{1})_{\perp} \) \( B_{z} \). As a result, the contribution of these longitudinal terms into SHG can always be neglected, and hence, the component of dc magnetic field normal to the direction of the wave propagation \( k_{1} \) is the only one that contributes to SHG. It is worth noting that in the case of a homogeneous magnetic field (see Sec. 3.2), there is no need even for this assumption, since in that case \( B_{y} = 0 \) and \( \partial B_{z}/\partial z = 0 \). Therefore, the case of homogeneous dc magnetic field presents sort of a test ground for the theory of the SHG effect.

It is well known in optics and quantum electronics that for the case of small diffraction, the Maxwell's equations [see e.g. Eqs. (3) and (4) in the case of linear media] can be reduced to a so-called paraxial wave equation for the “slow” component of linear field \( E_{1}, u_{1} \), (where \( u_{1} \) is determined by \( E_{1} = u_{1} \exp \left[i(k_{1}y - \omega_{1}t)\right]\)):

\[
2i k_{1} \frac{\partial u_{1}}{\partial y} + \Delta_{\perp} u_{1} = 0 ,
\]

where \( \Delta_{\perp} = \partial^{2}/\partial x^{2} + \partial^{2}/\partial y^{2} \) is “transverse” Laplacian.
The general solution of this equation can be presented as a sum of so-called Gaussian modes (or beams). For the sake of simplicity, here, we will only consider a linearly polarized fundamental Gaussian beam (TEM$_{00}$ mode) which is cylindrically symmetric in the plane $x, z$. In this case, $\Delta \lambda = \partial^2 / \partial r^2 + r^{-1} \partial / \partial r$, where $r = (x^2 + z^2)^{1/2}$ is the distance from the center of the beam in the plane $x, z$; due to cylindrical symmetry of the fundamental Gaussian beam $E_1$ does not depend on the polar angle, $\tan^{-1} (x/z)$. The solution for electric field $E_1$ and magnetic induction $B_1$ of such a beam then is

$$E_1 = E_1 (\sin \theta_1 \hat{e}_x + \cos \theta_1 \hat{e}_z) \frac{w_0}{w} \exp \{i \left[ k_1 y - \eta - \omega_1 t \right]$$

$$- \left[ r^2 \left( 1/w^2 - i k/2R \right) \right] \} ,$$

(18)

$$B_1 = \hat{e}_y \times E_1 ,$$

(19)

where $E_1 = \text{constant}$, $\theta_1$ is an angle between polarization of the fundamental wave and the $z$-axis. In Eq. (18), we introduced the following variables:

$$w^2(y) = w_0^2 \left( 1 + y^2/y_0^2 \right) ,$$

(20)

$$R(y) = y \left( 1 + y_0^2/y^2 \right) ,$$

(21)

$$\eta(y) = \tan^{-1} (y/y_0) ,$$

(22)

where $w(y)$ is a radius of the beam determined by the condition $|E_1(r=w)|/|E_1(r=0)| = 1/e$; $w_0$ is the minimum radius of the beam, i.e. the radius at the beam waist, $y=0$; $R(y)$ is a radius of curvature of the phase front of the beam; $\eta(y)$ is a phase of the wave at the center of the beam i.e. at $r=0$; $y_0 = \pi w_0^2 / \lambda_1$ is the half of the confocal parameter (i.e. the Rayleigh parameter or diffractional length) of the laser beam with $\lambda_1 = 2 \pi/k_1$. Taking the gradient of the first equation of Eq. (12) on both sides, eliminating $\nabla \times B_2$ and using the second equation of Eq. (12), one obtains the wave equation for SHG:

$$- \nabla^2 E_2 + (1/c^2) \partial^2 E_2/\partial t^2 = - (1/c) \partial \left\{ F(y, r) \cdot \exp \left\{ 2i \left( k_1 y - \omega_1 t \right) \right\} \right\} / \partial t .$$

(23)

Assuming the SHG field $E_2$ in the form of

$$E_2 = u_2 (y, r) \exp \left\{ 2i(k y - \omega t) \right\} ,$$

(24)

where $u_2$ is time-independent amplitude of SHG, and using the same standard paraxial approximation as the one which produces a Gaussian solution for the fundamental wave, Eq. (18), (i.e. presuming that $k_2 |\partial u_2/\partial y| \gg |\partial^2 u_2/\partial y^2| \ll k_2^2 |u_2|$, where $k_2 = 2k_1 = 4 \pi/\lambda_1$), we obtain the paraxial wave equation for $u_2$:
\[ 4ik_1 \frac{\partial u_2}{\partial y} + \Delta \perp u_2 = -2ik_1 F(y, r), \tag{25} \]

where again \( \Delta \perp = \frac{\partial^2}{\partial r^2} + r^{-1} \frac{\partial}{\partial r} \).

Now we can calculate SHG in a Gaussian beam and, in general, inhomogeneous dc magnetic field using only transverse dc field \( B_\perp = B_\perp(y) \hat{e}_z, B_\perp(y) = B_\perp(y) \), where at this point the function \( B_\perp(y) \) is not specified yet. Using the expressions for the laser (i.e. fundamental) beam, Eqs. (18), (19) and (13) for the driver \( F \), we obtain the driving term for Eq. (25):

\[ -2ik_1 \cdot F = 2ik_1 \xi E_1^2 B_\perp \left[ (4 \sin^2 \theta_1 + 7 \cos^2 \theta_1) \hat{e}_x - 3 \sin \theta_1 \cos \theta_1 \hat{e}_z \right] \]
\[ \times \frac{w_0^2}{w^2} \left[ \frac{w'}{w} - \frac{B'_\perp}{2B_\perp} + i\eta' - \frac{r^2}{w^2} \left( 2 \frac{w'}{w} - i \frac{k_1 w^2 R'}{2R} \right) \right] \]
\[ \times \exp \left[ 2i(k_1 y - \eta - \omega t) - 2r^2 \left( 1/w^2 - ik_1/2R \right) \right], \tag{26} \]

where prime denotes a derivative with respect to the axis of propagation, i.e. \( d/dy \).

One can immediately note that in the limiting case of a plane fundamental wave \( (w' = 0, \eta' = 0, R' = 0) \) and homogeneous dc field \( (B'_\perp = 0) \), the driver, Eq. (26), vanishes, which is consistent with the conclusion of Sec. 2, Eq. (15).

The driving term Eq. (26) can essentially be broken down into two components, one of them independent of \( r \), and the other proportional to \( r^2 \). Therefore, it is natural to seek for a solution for \( u_2 \) in Eq. (25) in the form of the Gaussian beam, Eq. (18), in which, however, pre-exponential scalar factor now consists of two respective components, \( S(y) + Q(y) r^2/w_0^2 \), instead of only \( r \)-independent component \( w^{-1}(y) \) as in fundamental Gaussian beam, Eq. (18). Here, \( S(y) \) is the relative amplitude of the main Gaussian component of SHG, whereas \( Q(y) \) is the relative amplitude of a so-called “doughnut” Gaussian component. Indeed, presuming \( u_2 \) in Eq. (25) in the form of

\[ u_2 = \xi E_1^2 [S(y) + Q(y) \cdot (r^2/w_0^2)] \exp \left[ -\xi_2 (y)r^2/2 \right] \]
\[ \times \left[ (4 \sin^2 \theta_1 + 7 \cos^2 \theta_1) \hat{e}_x - 3 \sin \theta_1 \cos \theta_1 \hat{e}_z \right], \tag{27} \]

where \( S, Q \) and \( \xi_2 \) are unknown functions of \( y \), substituting \( u_2 \) in the form of Eq. (27) into Eq. (25), we obtain the following differential equations for \( S \) and \( Q \):

\[ 4ik_1 S'' - 2S' \xi_2 + 4Q/w_0^2 \]
\[ = + 2ik_1(w_0^2/w^2) B_\perp(w'/w - B'_\perp/2B_\perp + i\eta') \exp (-2i\eta), \tag{28} \]
\[ 4ik_1 Q' - 6Q \xi_2 = (i/2) k_1 B_\perp \cdot \xi_2 \cdot (w_0^2/w^2) \exp (-2i\eta), \tag{29} \]
where

\[ \xi_2 = 4(1/w^2 - ik_1/2R), \]  

(30)

which follows form comparison of Eqs. (27) and (26). Equations (28) and (29) can readily be solved as

\[ S(y) = s \exp(-i\eta)/(1 + y^2/y_0^2)^{1/2}, \]  

(31)

\[ Q(y) = q \exp(-3i\eta)/(1 + y^2/y_0^2)^{3/2}, \]  

(32)

where \( s \) and \( q \) are

\[ s(y) = \frac{1}{4} \left\{ (w_0/w) \exp(-i\eta) \right\} \left\{ -B \downarrow(y) + i (w_0/y_0) \int_{-\infty}^{y} w^{-1} \exp(-i\eta) B \downarrow(y) \, dy \right\} \]  

(33)

and

\[ q(y) = -(1/2) (w_0/y_0) \int_{-\infty}^{y} w^{-1} \exp(-i\eta) B \downarrow(y) \, dy \].  

(34)

Integration constants implicitly incorporated in Eqs. (31) and (32) by the choice of the lower limit of the integrals in Eqs. (33) and (34), have been determined using the "boundary" condition in infinity, whereby the photon flux is zero at \( y = -\infty \):

\[ P_2 \big|_{y = -\infty} = \int_{0}^{\infty} \frac{|E_2|^2}{754} 2\pi \sigma d\eta \big|_{y = -\infty} = 0. \]  

(35)

\(|E_2|\) is determined from Eqs. (27), (31) and (32) as

\[ |E_2| = \xi E_1^2 \cdot \frac{w_0}{w} |s + q \cdot \frac{r_2}{w_0^2} e^{-2\eta/2} e^{-2r_2^2/2w_0^2} (33 \cos^2 \theta_1 + 16)^{1/2}. \]  

(36)

As mentioned, the term proportional to \( q \), is a doughnut component, i.e. conical radiation. The conical angle of its propagation with respect to the \( y \)-axis is calculated from Eq. (36) as \( \phi_c = \lambda_1/\sqrt{2} \), \( \pi w_0 = \phi_d/\sqrt{2} \), where \( \phi_d \) is a diffraction angle of the fundamental beam, i.e. as might be expected, it depends only on geometrical configuration, not on the amplitude of the fundamental field.

Substituting Eq. (36) into Eq. (35), and integrating it, one obtains the photon flux at any point at the \( y \)-axis:

\[ P_2 = 754 \xi^2 \frac{P_1^2}{\pi w_0^3} (33 \cos^2 \theta_1 + 16) T(y), \]  

(37)
where \( P_1 = \pi \omega_0^2 E_1^2 / 1508 \) is the photon flux at the fundamental frequency and factor \( T \) is

\[
T = |s + \frac{1}{4} q e^{-2s^2} j^2 + \frac{1}{16} | \mathbf{q} |^2. \tag{38}
\]

A few general properties of the discussed effect could be addressed using general results obtained in this section for a Gaussian beam in a dc magnetic field with arbitrary distribution. One of the issues of interest is polarization properties of the magnetized vacuum as a nonlinear medium. Because of the spatial anisotropy due to dc magnetic field, SHG should depend upon the polarization of the laser fundamental wave. The polarization angle of the SHG defined in the similar form as that for the fundamental wave can be calculated using Eq. (27):

\[
\theta_2 = \tan^{-1} \left[ (3 \cos 2 \theta_1 + 11) / 3 \sin 2 \theta_1 \right], \tag{39}
\]

(see Fig. 2). It follows from Eq. (39) that the ratio of maximum intensity of the second harmonic (which occurs at \( \theta_1 = 0 \)) to minimum intensity (at \( \theta_1 = \pi / 2 \)) is \((7/4)^2 = 3\) (see Fig. 2a), which can be directly measured in an experiment. Note that the generated second harmonic can never be polarized along the direction of the dc magnetic field, i.e. the polarization angle \( \theta_2 \neq 0 \) in Eq. (39). A minimal angle between the polarization of the second harmonic and the dc magnetic field is \( (\theta_2)_{\min} = 74 \) deg, and the corresponding polarization angle of the fundamental wave is \( \theta_1 = \theta_{1v} = 53 \) deg, i.e. there is a prohibited sector for the polarization of the second harmonic, \(- (\theta_2)_{\min} < \theta_2 < (\theta_2)_{\min}\), see Fig. 2b. All these polarization properties can be used in a future experiment to rule out all other (i.e. nonvacuum) nonlinear mechanisms.

Another issue of interest is the comparison of our general result for SHG in vacuum with that in conventional nonlinear materials. It is also associated with the understanding of the significance of the doughnut component of SHG which might be closely related to the mechanism of nonvanishing contribution of box diagram in the Gaussian beam case, in contrast to the plane wave configuration. For this purpose, it is instructive first to consider the case of an inhomogeneous dc magnetic field with a symmetric spatial distribution, and in the situation in which the center of the dc magnetic field coincides with the position of the beam waist. One can see from Eq. (33) that in this case, after passing through the entire region of the magnetic field, the coefficient \( s \) vanishes, \( s(+ \infty) = 0 \), which means that SHG due to the main Gaussian component of generated beam vanishes here. This is consistent with the conventional nonlinear optics result that in second-order classical nonlinear medium, a Gaussian TEM\(_{00}\) fundamental beam cannot produce a nonzero SHG if its waist point is located in the middle of the nonlinear medium.\( ^{20} \) We will also see in the Secs. 3.1 and 3.2, where the results are found for specific configuration of dc magnetic field that even in the case when the main Gaussian component of SHG
Fig. 2. (a) The normalized intensity $I/I_0$ for the second harmonic with polarization angle $\theta_2$ with respect to the dc magnetic field [see inset in (b)] vs the polarization angle $\theta_1$ (in deg) of a linearly polarized fundamental wave (solid line); (b) the polarization angle $\theta_2$ (in deg) of the second harmonic vs the polarization angle $\theta_1$ (in deg) of a linearly polarized fundamental wave; the second harmonic is never polarized in sectors $|\theta_2 - 90^\circ| > 16^\circ$. Inset: The wave propagation configurations for both the fundamental wave and second harmonic with respect to the dc magnetic field $B_0$. 
is nonvanishing, its contribution compared to that of doughnut component is
insignificant. The dominant doughnut component is essentially a conical radiation
(in the infinity). This component may have something to do with a possible
explanation of nonvanishing contribution of box diagram, in terms of improved
phase matching for SHG. One can argue that since a Gaussian beam of
fundamental radiation has a finite angular spectrum, the SHG radiation chooses
only certain nonzero angles of its propagation at which the phase-matching
condition (which essentially corresponds to conservation of momentum of the
system of the two Gaussian beams) could be satisfied. For that component, there
are no SHG photons propagating strictly along the y-axis (i.e. the axis of
propagation of fundamental radiation); the intensity of that component in the
center of the SHG beam is always zero.

3.1. Homogeneous dc Magnetic Field

In this section, we calculate SHG in a Gaussian fundamental beam and
homogeneous dc magnetic field. Consider a laser beam propagating along the y-axis
in a dc magnetic field homogeneous in a limited region,

\[ B_\perp = B_0, \quad -L_{\text{in}} < y < L_{\text{out}}. \]  \hspace{1cm} (40)

The laser beam enters the system at the point \( y = -L_{\text{in}} \), exits at the point \( y = L_{\text{out}} \);
the origin, \( y = 0 \), is located at the waist of the Gaussian beam, see Fig. 3a.
Substituting Eq. (40) into Eqs.(33) and (34), and using the condition \( dB_0/dy = 0 \), we
obtain explicit expressions for \( s \) and \( q \):

\[ s = B_0 \left[ -1 + \frac{1}{2} \ln \left( 1 + y^2/y_0^2 \right) + itan^{-1} \left( y/y_0 \right) \exp \left( -i \eta \right) \right] \exp \left( -i \eta / 4 \right) \left( 1 + y^2/y_0^2 \right)^{1/2}, \]  \hspace{1cm} (41)

\[ q = -\frac{1}{2} B_0 \left[ itan^{-1} \left( y/y_0 \right) + \frac{1}{2} \ln \left( 1 + y^2/y_0^2 \right) \right]. \]  \hspace{1cm} (42)

At the "edges" of magnetic field (i.e. around \( y = -L_{\text{in}}, y = L_{\text{out}} \)) the dc magnetic
field is smoothly vanishing, but we will assume that the characteristic distance of this
vanishing is much smaller than that of \( \psi \), i.e. smaller than the diffraction parameter
\( y_0 \). Thus, the calculation of the integration in Eqs.(33) and (34) is readily simplified
by presuming \( B_\perp = \text{constant} = B_0 \) and performing the calculation with the lower
limit of integration being \( -L_{\text{in}} \) instead of \( -\infty \). This procedure is valid if, e.g. the
distance between magnetic plates \( d \), is much smaller than the confocal parameter \( y_0 \),
i.e. \( d \ll y_0 \), which is easily attained under realistic experimental conditions. If the
waist of the laser Gaussian beam is positioned sufficiently deep in the dc magnetic
field, i.e. when \( y_0 \ll L_{\text{in}}, L_{\text{out}} \), our results for homogeneous dc field with the
boundary condition, Eq. (35), coincide with that of exact calculations for the case of
smoothly varying inhomogeneous dc magnetic field of dipole, see Sec. 3.2.
Fig. 3. The laser beam propagating (a) in the homogeneous dc magnetic field, and (b) in the inhomogeneous dc field of a 2-D magnetic dipole.
As mentioned and seen from Eq. (36), the SHG field, in general, consists of two components. One of them (the main component) is of the same type as the fundamental (laser) Gaussian beam (i.e. the term $S(y)$ in Eq. (27) or $s$ in Eq. (36)), and another is the doughnut component (the term $Qr^2/w_0^2$ in Eq. (27) and $qr^2e^{-2'y}/w^2$ in Eq. (36)). It can be shown using Eqs. (37), (38), (41) and (42), that in most of the cases of interest, the doughnut component is dominant, and its contribution to the photon flux (see Eq. (38)) exceed the one corresponding to the main component by order of magnitude or more. Therefore, this property of the intensity profile for SHG can be used in the experiment for separating the SH photons from the fundamental photons spatially, and therefore, the sensitivity of the measurement can be somewhat enhanced. Let us consider how the combined power of both those components (or their total photon flux), Eqs. (37) and (38), depends on the experimental arrangement for the laser beam inside the magnetic field, which will help us choose the optimal geometry of a possible experiment.

First, let us consider the arrangement whereby the waist of the laser beam is positioned deep inside the magnetic field, i.e. $L_{\text{in}}, L_{\text{out}} \gg y_0$. The result for $L_{\text{in}}/y_0 = 100$ is depicted by curve 1 in Fig. 4. If $y \gg y_0$ (or $L_{\text{out}} \gg y_0$) the contribution of the main component (the term $S(y)$ in Eq. (27)) into the SHG vanishes. As mentioned above, this property that the main SHG component to vanish if $L_{\text{in}}, L_{\text{out}} \gg y_0$ is well known in the theory of SHG in "conventional" second-order nonlinear optics. Thus, this SHG component could be regarded as "classical" SHG process, whereas the doughnut component is a feature peculiar to vacuum. The number of generated photons reaches a maximum inside the dc magnetic field at the waist of the beam, see curve 1 in Fig. 4, with the factor $T$ in Eq. (37) becoming

$$ (T)_{\text{max}} = \frac{B_0^2}{128} \left[ \ln \left( \frac{L_{\text{in}}}{y_0} \right) \right]^2. \quad (43) $$

However, at the output face of the magnetic field (if $L_{\text{in}}, L_{\text{out}} \gg y_0$), the total number of SH photons decreases and tends to some constant with $T$ in Eq. (37) as

$$ T = |q|^2/8 = \frac{\pi^2}{32} B_0^2. \quad (44) $$

Once again, in this case $s = 0$, i.e. the result, Eq. (44), corresponds to the only surviving doughnut component. As the entrance face approaches the beam waist, the SHG power at the waist ($y = 0$) decreases, see curve 2 for the case $L_{\text{in}} = 10$ $y_0$ in Fig. 4.

For the fixed length of the homogeneous magnetic field, $L_{\text{hom}} = L_{\text{out}} + L_{\text{in}}$, the number of SH photons reaches maximum when the waist of the beam is positioned right at either the output or input face of the magnetic field, i.e. when either $L_{\text{in}} = 0$ or $L_{\text{out}} = 0$. For such an arrangement, the term $T$, Eq. (38) (and the total SHG power) reaches maximum:
Fig. 4. Relative power of SHG vs the relative position along the propagation axis \( y/y_0 \) for different values of \( L_{in}/y_0 \): curve 1 - \( L_{in}/y_0 = 100 \) (the beam waist is deep inside the dc magnetic field); curve 2 - \( L_{in}/y_0 = 10 \); curve 3 - \( L_{in}/y_0 = 0 \) (the field entrance point is right on the beam waist); curve 4 - \( L_{in}/y_0 = -5 \) (the beam waist is outside of the magnetic field).

\[
(T)_{\text{max}} = \frac{B_0^2}{128} [\ln (L_{\text{hom}}/y_0)]^2 ,
\]  

(45)

see e.g. curve 3 in Fig. 4 for \( L_{in} = 0 \).

For the case in which the focal point (i.e. the waist position) is located outside the dc magnetic field, i.e. \( L_{in} \gg -y_0 \), the curve depicting SHG behavior for \( L_{in} = -5y_0 \) is shown as curve 4 in Fig. 4. We can see that the SHG power significantly reduces compared to the above cases.

3.2. **Inhomogeneous Magnetic Field: A Magnetic Dipole**

In the previous section, we discussed SHG induced by a laser beam propagating in a homogeneous dc field of a finite extent for the case in which the “wings” of the dc field vanish sufficiently rapidly along the axis of propagation, but remains constant within some “length of homogeneity”, \( L_{\text{hom}} = L_{in} + L_{out} \). In this section, we
will find an exact (in the framework of the paraxial approximation) solution for SH generated by a Gaussian laser beam propagating in a dc magnetic field which is substantially inhomogeneous along the entire length of the nonlinear interaction, and has a realistic and explicitly expressed spatial distribution of an infinite extent. The limiting result of these calculations for SHG for the case in which the spatial scale of dc field tends to infinity coincided with that obtained by us in the previous section for the case \( y_0 \ll L_{\text{in}}, L_{\text{out}} \).

Equations (33) and (34) are valid for an arbitrary spatial distribution of the transverse dc magnetic field along the axis of propagation, \( B_\perp(y) \). For specific calculations, we choose a simple and yet fundamental inhomogeneous magnetic field originated by two thin parallel "magnetic" wires positioned normally to the laser beam (and extended, e.g. along the \( x \)-axis) with the distance \( d \) between them, Fig. 3b. (It is well known that the same field can also be originated by a pair of cylinders of finite radius \( r_c \). In such a case \( d = \sqrt{d_z^2 - 4r_c^2} \), where \( d_z \) is the distance between centers of the cylinders.) The magnetic field of such a 2-D dipole is

\[
B_x = 0; \quad B_y = \frac{B_0 d}{4} \left[ \frac{y - L_f}{(y - L_f)^2 + z^2} - \frac{y - L_f}{(y - L_f)^2 + (z - d)^2} \right], \quad (46)
\]

\[
B_z = \frac{B_0 d}{4} \left[ \frac{z}{(y - L_f)^2 + z^2} + \frac{d - z}{(y - L_f)^2 + (z - d)^2} \right], \quad (47)
\]

where \( B_0 \) is the dc field at the origin \( (y = z = 0) \), and \( L_f \) is the distance between the center of the magnetic field and the position of the waist of the fundamental Gaussian beam, see Fig. 3b. In particular, in the plane of symmetry, \( z = d/2 \), we have

\[
B_\perp = B_z = \frac{B_0}{1 + 4 \{(y - L_f)^2 / d^2\}}, B_y = 0. \quad (48)
\]

Substituting Eq. (48) into Eq. (34), we can simplify Eq. (34) as

\[
q = - \frac{id^2 B_0^2}{8} \frac{1}{2L_f y_0 + i(L_f^2 + d^2/4 - y_0^2)} \left\{ - \frac{1}{2} \ln \left[ \frac{(y - L_f)^2 + d^2/4}{y^2 + y_0^2} \right] + i \left[ \tan^{-1}\left( \frac{y}{y_0} \right) + \pi/2 \right] - \frac{y_0 + iL_f}{d/2} \left[ \tan^{-1}\left( \frac{y - L_f}{d/2} \right) + \pi/2 \right] \right\}. \quad (49)
\]

Consider the case in which the fundamental beam passes through the entire magnetic field, i.e. \(-\infty < y < \infty\). Equation (49) can then be simplified as

\[
|q(y \to \infty)|^2 = \frac{B_0^2 \pi^2 d^2/16}{L_f^2 + (y_0 + d/2)^2}. \quad (50)
\]
The value \( |q(y \rightarrow \infty)|^2 \) reaches its maximum when \( L_f = 0 \), i.e. if the waist of the fundamental Gaussian beam coincides with the center of the magnetic field, \( y = 0 \). Therefore,

\[
|q(y \rightarrow \infty)|^2_{\text{max}} = \frac{B_0^2 \pi^2}{4} \left( \frac{d}{d + 2y_0} \right)^2.
\]

(51)

The transition to a homogeneous field would correspond to the limiting case \( d/y_0 \rightarrow \infty \) which results in \( |q|^2_{\text{max}} = (B_0 \pi/2)^2 \), in which case

\[
T = |q|^2_{\text{max}}/8 = \frac{\pi^2}{32} B_0^2.
\]

(52)

This is exactly the same as the result, Eq. (44), for a homogeneous field with \( L_{\text{in}}, L_{\text{out}} \gg y_0 \), as expected.

4. Comparison Between Box and Hexagonal Diagram Contributions

All the calculations in this paper have been done in the QED box approximation involving only term \( L_4 \) in Eq. (1). As is mentioned in the introduction and in Sec. 2, the next, hexagonal term \( L_6 \) in Eq. (1) makes much smaller contribution to the process for any field accessible in the laboratory now and in the foreseeable future. Let us consider the order of magnitude of the hexagonal term. It can be shown that using e.g. Refs. 1 and 2, the nonlinear terms in hexagonal approximation can be expressed as

\[
\mathbf{D}^{\text{NL}} = \gamma \left\{ [(E \cdot B)^2 + 6 (E^2 - B^2)^2/13] \mathbf{E} + (E^2 - B^2) (E \cdot B) \mathbf{B} \right\}
\]

(53)

and

\[
\mathbf{H}^{\text{NL}} = \gamma \left\{ [(E \cdot B)^2 + 6 (E^2 - B^2)^2/13] \mathbf{B} - (E^2 - B^2) (E \cdot B) \mathbf{E} \right\},
\]

(54)

where \( \gamma = 26\alpha/315 \pi B_{\text{cr}}^4 \). Presuming at this point that inhomogeneity plays the same role in both box and hexagonal approximations, the ratio of the total SHG power for these cases can be expressed as

\[
\frac{(P_2)_{\text{hex}}}{(P_2)_{\text{box}}} \sim 14 \left[ \frac{B_0}{B_{\text{cr}}} \right]^4.
\]

(55)

For the value of dc magnetic field used in this paper, this ratio is \( \sim 10^{-24} \). Presuming on the other hand that in the worst case scenario, the hexagonal nonlinearity produces SHG even for plane wave (which is not true for the box diagram), our crude estimate shows that this ratio is still smaller than \( \sim 10^{-16} \) for the parameters used in this paper.
5. Experiment-Related Considerations

Whether the nonlinear optical effect in vacuum considered here could be experimentally observed, depends on the technology available now or in the near future. There are a few components of the experiment that seem to be crucial: high-power lasers, photon-counting systems, dc (or pulsed) magnets, and vacuum systems. In this section, we shall estimate the experimental parameters required to observe the vacuum SHG. The state-of-the-art photon counting systems provide a dark photon count rate $r_{\text{dark}} \sim 10$ photons/s and a typical quantum efficiency $\eta \sim 0.25$.\(^{21}\) For the ideal spectral filtering and provided that by using gating, the detector is open only during the laser pulse,\(^{21}\) the signal-to-noise ratio is $\text{SNR} = \langle n_{\text{det}} \rangle / \langle n_{\text{dark}} \rangle$, where $\langle n_{\text{det}} \rangle = \eta \langle n_{\text{SHG}} \rangle$ is the averaged number of detected photons per pulse, $n_{\text{SHG}}$ is the number of SHG photons generated per pulse, $\langle n_{\text{dark}} \rangle = r_{\text{dark}} \tau_1$, and $\tau_1$ is the duration of a laser pulse. Stipulating now that $\text{SNR}$ is e.g. unity, considering the case of the normal polarization ($\theta_1 = \pi/2$, which would correspond to the minimal SH photon output), using Eqs.(37) and (44) we obtain the critical laser peak power $P_{\text{cr}}$ required for such $\text{SNR}$:

$$P_{\text{cr}} = 9 \times 10^{21} y_0^{1/2}/B_0,$$

(56)

where $y_0$ is in cm, $B_0$ in Gauss, and $P_{\text{cr}}$ in watts. With pulsed magnet technology,\(^{22}\) the best parameter values are $B_0 \sim 15 \times 10^6$ Gauss, bore diameter $d \sim 0.7$ cm and pulse duration $\sim 10^{-6}$ s.

The development of new powerful lasers and pulsed magnetic fields is proceeding at a rapid pace, and a pulse energy of 1 MJ–10 MJ, pulse width of 10 ns–20 ns, and repetition rates of 10 Hz,\(^{23}\) as well as possible generation of a magnetic field $\sim 10^8$ Gauss with a pulse duration of $10^{-9}$ s and bore diameter of 0.1 cm using high-power lasers,\(^{24}\) may be only a few years away. However, the laser peak power required to satisfy Eq. (56) can be achieved using existing systems. For an order-of-magnitude estimate, we assume a diffraction limited focusing of the fundamental laser beam such that the confocal of the laser beam $2y_0$ is equal to $10\lambda$. A high-power pulsed Nd:Glass laser with either $\lambda_1 \sim 0.35$ $\mu$m (NOVA\(^{25}\)) or $\lambda_1 \sim 0.53$ $\mu$m (GEKKO XII\(^{26}\)), $\tau_1 \sim 10^{-9}$ s, can provide the laser energy of 6 KJ/pulse–10 KJ/pulse in each of 10–12 beam lines. For this energy, the laser peak power
d
required to satisfy Eq. (56) for the normal polarization ($\theta_1 = \pi/2$, the worst case) even within a single laser-beam is $\sim 10^{13}$ watts. Recently, efforts have been made to increase both the output power (intensity) of very short pulse lasers and their repetition rate (which can be as high as 1 Hz–10 Hz). For XeCl,\(^{27}\) Nd:Glass,\(^{28}\) and KrF\(^{29}\) lasers, the required laser peak power to satisfy Eq. (56) is $\sim 10^{13}$ watts.

Since for all the high-power laser systems available, $\langle n_{\text{det}} \rangle$ is very small, one has to use averaging of photon counts over a few (usually incoherent) laser-beam lines for a single pulse or/and over many pulses of laser. Assuming a Poisson distribution of SHG photons,\(^{21}\) the probability of not observing any SHG photons within $N$ laser pulses is $p = \exp(-N\langle n_{\text{det}} \rangle)$. Stipulating again that $p$ should be sufficiently small, e.g.
\( N \geq \ln 10 / \langle n_{\text{det}} \rangle = 2 \times 10^{43} y_0 \tau_1 / J_1^2 B_0^2 \),

(57)

where \( \tau_1 \) is in s and \( J_1 \) is the energy of the fundamental laser beam per pulse or beam in joule. Our estimate shows that the number of pulses required to satisfy Eq. (57) is \( \sim 10^{14} \) using Refs. 27–29 and therefore, not yet achievable using existing laser systems.

The SHG attributed to residual atoms in the laboratory vacuum system can mask the photon-photon scattering effect. In order to make an order-of-magnitude estimate of the contribution of these residual particles and to evaluate the vacuum pressure necessary to rule out nonvacuum components in the SHG, we presume that nonvacuum nonlinearity is mainly attributed to the plasma of magnetized free electrons, since at the required laser intensities, the residual gas is expected to be almost fully ionized. The nonmagnetized free electrons under the action of intense laser radiation can generate only odd-order higher harmonics; the second-order harmonics can only be originated in the presence of the dc magnetic field. The numbers of second-harmonic photons per laser pulse, \( N_{\text{SHG}} \), for the normal polarization can be obtained using

\[
N_{\text{SHG}} = 4 \pi \varepsilon_0^4 \alpha \omega_0^2 \tau_2 E_{4}^2 n_e y_0 / \omega_1^7 m_e^2 c^2,
\]

(58)

where \( n_e \) is the number of free electron per unit volume and \( \omega_0 = e H_0 / m_e c \) is a cyclotron frequency of the electron. In good vacuum system, the residual gas consists basically of hydrogen atoms. Assuming therefore that each atom is single-ionized, and that GEKKO XII laser (\( \lambda_1 \sim 0.53 \mu m \)) and the highest pulsed magnetic field (\( \sim 15 \cdot 10^5 \) G) are used, we estimate that dark photon and free-electron contributions to the SHG become equal at a pressure \( \sim 4 \times 10^{-19} \) torr. Such high vacuum is not yet achievable. However, our calculation showed that for the (ideal) parallel polarization, \( \theta_1 = 0 \) (which corresponds to the maximum output of SHG photons), \( N_{\text{SHG}} = 0 \). In an unideal case, there still will be normal component of laser field present due to fluctuations of polarization. Assuming these fluctuations to be, say of \( \sim 10^{-4} \), i.e. \( |E_{4} / E_{1}|^2 \sim 10^{-4} \), we estimate, using Eq. (58) and noting that \( N_{\text{SHG}} \) depends on \( E_{4}^2 \), that the required pressure in such a case is \( 4 \times 10^{-11} \) torr which is quite accessible using the existing vacuum systems.

One of the ways to eliminate masking effects associated with possibility of SHG in laser amplifiers and other optical components (such as lenses and vacuum chamber windows), is to use sum frequency effects (instead of the second harmonic) by employing two lasers with different frequencies. A more detailed evaluation of all these processes could be made only at the design stage of a particular experiment.

A very large magnetic field could exist in many astronomical objects (e.g. in white dwarfs, where the spectral lines of elements still exist in the optical range); the
possibility exists that a second-harmonic signal generated by some characteristic spectral lines may be observed, and used to study the nonlinearity of the vacuum and intrinsic properties of stars.

6. Conclusion

We have discussed the feasibility of second-harmonic generation by the intense laser radiation in a vacuum which is due to photon-photon scattering in a dc magnetic field in the “box” diagram approximation. We believe that although the current state of technology still prevents immediate observation of this effect, rapid developments in the laser and magnet technology will allow for its observation in near future.

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