Amplification, Instability, and Chaos in Nonlinear Counterpropagating Waves

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Abstract

We demonstrate that two linearly polarised counterpropagating waves in a Kerr nonlinear medium with linear dispersion can exhibit amplification and multi-mode temporal instability, which are attributed to the combined effect of nonlinear index grating and linear dispersion.

1. Introduction

The cross-interaction of two counterpropagating laser beams [1]-[6] in a third order nonlinear material is a conventionally simple and fundamental process in nonlinear optics. The steady states of this interaction have been shown to exhibit various interesting and complicated features. Earlier theoretical investigations have demonstrated the existence of temporal instability and chaos under various extensions of this system, in particular, for a nonlinear medium having finite relaxation time [6]. However, the relaxation of nonlinear refractive index may not be the most likely mechanism of instability of beam intensity in most of the nonlinear systems since it imposes too stringent requirements on relaxation time.

Recently, we showed [7] that another physical factor, namely regular linear dispersion (i.e. frequency dependence of refractive index) can be a natural and universal agent for amplification of perturbations and temporal instability in the counterpropagating nonlinear waves. In this paper, we present the study of the effect and also explore its possible use for large broad-band amplification that may exist in the system if the pumping is below the threshold of instability. It is well known that linear dispersion can give rise to nonlinear optical effects such as spatial instability of single plane wave [8], formation of soliton in nonlinear fiber [9], and amplification of copropagating waves [10]. This suggests that linear dispersion in combination with nonlinear processes can dramatically change the dynamical behavior of a system.

2. Wave equations

In order to demonstrate the instability effect, we investigate the simplest spatially stable eigenpolarisation [2,3] whereby the two waves are linearly polarised with electric fields parallel to each other. (Spatial stability of the eigenpolarisation is essential since spatially unstable polarisation arrangements may result in temporal instability even if no other mechanism is present [11].)

![Figure 1. Counterpropagating waves in a Kerr nonlinear medium.](image)

For the configuration shown in Fig. 1, the total complex electric field of the two counterpropagating plane waves in the Kerr nonlinear medium is represented as: $\mathbf{E} = E_1(z,t)e^{i\omega z} + E_2(z,t)e^{-i\omega z}$, where $E_1(z,t)$ and $E_2(z,t)$ are the slowly varying envelopes of forward (+z) and backward (-z) propagating waves respectively. The dynamics of these envelopes is governed by two coupled nonlinear Schrodinger equations [7]:

\[
\begin{align*}
\frac{i}{-1}\frac{\partial E_j}{\partial z} + \frac{1}{v_s}\frac{\partial E_j}{\partial t} - \frac{\mu}{2} \frac{\partial^2 E_j}{\partial t^2} &= -\beta(2I_{j-f} + I_j)E_j ; \quad j = 1, 2
\end{align*}
\]
where $I_j = |E_j|^2$ is the intensity of the respective propagating wave, $\mu = \frac{\partial^2 k}{\partial \omega^2}$ is the linear dispersion parameter, $n$ is the linear refractive index at the frequency $\omega$, $v_0$ is the linear group velocity, $n_2$ is the nonlinear refractive index coefficient, and $k$ is the wave number in the medium. The coefficient $2$ in the right-hand side in Eq. (1) reflects light-induced non-reciprocity [12]. The problem of temporal instability of cw wave propagation for certain signs of dispersion and nonlinearity is isomorphic to the problem of cross-induced self-focusing bistability [13] and spatial instability of plane counterpropagating waves in nonlinear medium [14], since the latter problem can be described by Eq. (1) in which the term with $\frac{\partial^2 E}{\partial x^2}$ is replaced by $\frac{\partial^2 E}{\partial z^2}$ where $z$ is a coordinate in the transverse plane (i.e. spatial dispersion).

3. Instability

To analyse the small perturbation stability of the steady state, we represent both waves in the system as slightly perturbed steady state in the form:

$$E_j = E_{j0}(\xi)[1 + A_{1,j}(\xi)e^{i\tau} + A_{2,j}(\xi)e^{-i\tau}]$$  (2)

where $j = 1, 2$, $E_{j0}$ is the steady solution for Eq. (1), $A_{1,j}$ and $A_{2,j}$ are normalised amplitude of the small perturbations, $\tau = v_0 L/\lambda$ is the normalised time, and $\xi = z/L$ is the normalised distance of propagation. Substituting Eq. (2) into Eq. (1) and linearizing the latter equation, we obtain four linear propagation equations for $A_{k,j}$:

$$\frac{dA_{k,j}}{d\xi} = i(-1)^k(2\lambda - d\lambda^2) + p_j A_{3-k,j}$$  (3)

where $p_j = \beta I_{j0} L$, $d = \mu v_0^2/2L$, and $I_{j0}$ is the intensity of each wave. Here $p_j$ are the normalised pumping intensities of the two beams (note that $p_j$ can have negative sign depending on nonlinearity $\beta$), and $d$ is the normalised dispersion. At this point we simplify our calculation, assuming equal intensities of the waves, i.e. $I_{10} = I_{20} = I$. Using the boundary conditions for the perturbation amplitudes $A_{k,j}$ for $k = 1, 2$ and $j = 1, 2$:

$$A_{1,1}(\xi=0) = A_{1,2}(\xi=1) = A_{2,1}(\xi=0) = A_{2,2}(\xi=1) = 0 ,$$

one can obtain an equation for $\lambda$. The boundaries of unstable regions are found by setting $\text{Re}(\lambda) = 0$ and solving for the normalised driving intensity $p = \beta I L$ and $\text{Im}(\lambda)$ for a given dispersion.

The numerical results of this procedure are depicted in Fig. 2 which shows the normalized threshold intensity $\rho_{cr} (>0)$ versus normalized dispersion $d$ in a semi-log plot. The solid line is the ultimate (encompassing) boundary of instability. Above this boundary there are numerous solutions corresponding to various unstable modes of oscillation. As the dispersion decreases, the boundaries for individual modes are too closely packed together to be shown in Fig. 2. These individual boundaries correspond to the longitudinal modes in a light-induced, distributed-feedback resonator. Only a few of these solutions and the boundary encompassing all the unstable solutions (i.e. the solid line) are shown in Fig. 2. In the inset of Fig. 2, the number of unstable modes is plotted against $|d|/\rho_{cr}$ where $\rho_{cr} = 1.98$ is the threshold intensity for the case $|d| = 0.01$. One can see that when the pumping is only slightly above the threshold, only one mode become unstable, then two, and so on. When the number of modes is greater than one, they compete with each other and some of them can become dominant while others fade out.

The (encompassing) threshold intensity increases as dispersion decreases; our numerical calculations show that for sufficiently small $|d|$, this dependence can be best described by a surprisingly simple relationship

$$\rho_{cr} = -\text{sgn}(pd) \log_\eta |d|$$  (4)

where $\eta$ is numerically determined to be $10.0 \pm 1\%$, and $\text{sgn}(pd)$ is the sign of $pd$. The necessary condition for initiating instability (i.e. to have $\rho_{cr} > 0$) is that the signs of nonlinearity $n_2$ and dispersion $\mu$ must be opposite, which coincides with the necessary condition for formation of a soliton and spatial instability in single-wave propagation.

To explore the full dynamical behavior in unstable region showed in Fig. 2, we have numerically integrated Eq. (1) with the dispersion fixed at $|d| = 0.01$ (we choose this unrealistically large
$|p/p_{cr}| = 1.02$ is shown in Fig. 3. At such a pumping intensity, only one mode is unstable (see the inset in Fig. 2), and eventually develops into periodic self-sustained oscillations with a stable amplitude. When the value of $|p/p_{cr}|$ is further increased to 1.28, the instability and resulting self-sustained oscillations develop much faster as is shown in Fig. 4. In this case three modes satisfy the excitation condition (see the inset in Fig. 2); as the oscillations develop in time, the second subharmonic is excited which later evolves into aperiodic oscillations with randomly modulated amplitude and phase which indicates onset of chaos. We expect this behavior to further develop into strongly pronounced chaos when the pumping intensity is significantly higher than the threshold.

In order to estimate the threshold intensity for instability, we consider an example of a 1 km long single-mode fiber with Ge-doped silica core at wavelength 1.55 μm with group velocity dispersion $D(k) = 6.5 \times 10^{-3}$ [note that $\mu = -D'(k)/(k^2)$] (the corresponding $d = -8.02 \times 10^{-16}$ m$^2$/W) [15], in the lossless approximation. We find the threshold intensity $I_{cr}$, Eq. (4), for such a fiber to be $9.3$ MW/cm$^2$, which is below the damage threshold $10^{-10}$ W/cm$^2$ for fused silica [16]. Crude estimate shows that the losses existing in the real fiber (~0.5 dB/km [15]) would require only about two times higher threshold intensity. If we use the SF-59 glass with $n_2 = 7 \times 10^{-15}$ cm$^2$/W [17] at wavelength 1.06 μm, and assume that $n_2$ at wavelength 1.55 μm is of the same order as that of 1.06 μm and that dispersion is roughly the same as for plain glass, the critical intensity is reduced to $\sim 425$ kW/cm$^2$ for the same length.

4. Amplification

To find the gain spectrum of this system, we inject a weak probe beam into the nonlinear medium and scan the beam frequency. In Fig. 5, we show a new configuration with two pump beams and four weak beams. Suppose that the probe beam $(E_{10}A_{11})$ has frequency of $\Omega+\delta\Omega$, i.e. its frequency deviated from that of the pump beams with normalized frequency $\Omega$ by $\pm\delta\Omega$; then we expect a phase conjugate signal $(E_{20}A_{22})$ to be reflected back with frequency $\Omega-\delta\Omega$. These two beams are referred to as direct-coupled waves [18]. The remaining two weak beams are referred to as cross-coupled forward $(E_{10}A_{21})$ and backward $(E_{20}A_{12})$ waves with their frequencies equal to $\Omega-\delta\Omega$ and $\Omega+\delta\Omega$ respectively. Since we consider a collinear geometry, the effect of the cross-coupled waves cannot be neglected [18].

Since the only differences between perturbation in Section 3 and the four signal waves in here are that the boundary conditions for one of the signal waves (specifically, for a probe wave) is nonzero, $\{A_{21}(\xi=0) = A_{12}(\xi=1) = A_{22}(\xi=1) = 0$, and $A_{11}(\xi = 0) = C = const\}$ and the detuning frequency is set by the probe wave, we can treat $A_{11}, A_{21}, A_{12}$, and $A_{22}$ in here using the same procedure outlined in Section 3. The numerical results of solving Eq.
Figure 5. Pump-probe configuration for amplification measurement where $E_1$ and $E_2$ are pumping waves and $E_{10}A_{1,1}$ is the input probe wave with $E_{10}A_{2,1}, E_{20}A_{1,2}$ and $E_{20}A_{2,2}$ generated as a result of the wave mixing process. (3) are shown in Fig. 6.

$$G_1 \approx \left\{ \begin{array}{ll} 1 + p^2 + 2(1+p^2/3) \left| dp \right| \delta \Omega^2 & \text{for } \delta \Omega << (\delta \Omega)_o \vspace{1ex} \\
\exp \left(2\delta \Omega [2 \left| dp \right| - (d \delta \Omega)^{21/2} \right) & \text{for } \delta \Omega > (\delta \Omega)_o \end{array} \right.$$

where $(\delta \Omega)_o = \left( \frac{p_d}{\left| dp \right|} \right)^{1/2} \ln(1 + p^2)$, $D$ is the determinant defining the threshold of instability [7]. Eq. (5) correctly predicts that the gain slowly raise from $p^2 + 1$, its value at $\delta \Omega = 0$, when $\delta \Omega < (\delta \Omega)_o$. This portion of the gain spectrum for $\delta \Omega << (\delta \Omega)_o$ corresponds to the contribution from dispersionless four wave mixing [19]. Beyond $(\delta \Omega)_o$, the variation of gain obeys $\sim \exp(2\delta \Omega [2 \left| dp \right| - (d \delta \Omega)^{21/2}])$ when pumping power $|p|$ is substantially below $p_{cr}$. This part of the gain is solely caused by dispersion-related process. The determinant $D$, which is equal to one in most conditions [7] (especially in the realistic situation when $|d| << 1$) except at near-threshold pumping, is responsible for the resonant modes and instability.

From Eq. (5), we note that a nonzero amplification (or $G_1 > 0$) can be obtained for any pumping intensity, however, for each fixed intensity $|p|$, the frequency detuning should be smaller than certain cutoff frequency, $(\delta \Omega)_{cr}$,

$$|\delta \Omega| = (\delta \Omega)_{cr} = \sqrt{2 \frac{|p|}{|d|}}.$$  

Therefore, a rough estimate of the amplification bandwidth is given by $(\delta \Omega)_{cr}$, Eq. (6), which is valid for arbitrary pumping, $|p| < p_{cr}$. The maximum bandwidth $(\delta \Omega)_{max}$ for a fixed dispersion $d$ is attained when $|p| = p_{cr}$, Eq. (4), i.e.,

$$(\delta \Omega)_{max} = \sqrt{2 \frac{\log_{10} |d|}{|d|}}.$$  

The location of the peak gain $(\delta \Omega)_{opt}$, i.e. the frequency of the mode with lowest threshold of instability, can be approximately estimated as:

$$(\delta \Omega)_{opt} = (\delta \Omega)_{cr}/\sqrt{2}$$

The results from these approximate equations for frequency, Eqs. (6) - (8), are qualitatively close to the exact values from numerical calculation shown in Fig. 6.

The results above indicate that the nonlinear fiber with dispersion, pumped by counterpropagating waves has great potential as an all-optical amplifier operating in cw or quasi-cw regime which may find application in optical gyroscopes, optical fiber communications, etc. One possible limiting factor in amplification is the losses in the fiber which have not been included in our calculation. To estimate the effect of losses in the optical fiber, we introduce a threshold intensity of amplification $I_{amp}$, which satisfies the following condition:

$$\alpha L = \Gamma(I_{amp})$$

where $\alpha$ is the loss coefficient and $\Gamma(I_{amp}) = \ln(G_1(I_{amp}))$ is the growth rate at the amplification threshold in the lossless approximation, i.e. the gain must be equal to loss at $I_{amp}$. Substituting Eqs. (5), (6), and (8) into Eq. (9), we
obtain a remarkably simple expression for $I_{\text{amp}}$ at
$(\delta n)_{\text{opt}}$:

$$I_{\text{amp}} = \alpha / (2n_2k)$$  \hspace{1cm} (10)

which does not depend on either length or dispersion. For the silica fiber
considered in Section 5, $\alpha = 0.5 \text{dB/km} = 0.12 \text{km}^{-1}$ [15]
and $n_2k = 1.3 \times 10^{-6} \text{km}^{-1}$ [16]. Hence, the threshold of
amplification is $46.2 \text{W/cm}^2$ (or 0.5% of the threshold of insta-
\begin{itemize}
\item [$\alpha$] The example with $L = 1 \text{km}$ and
\item [$\mu$] $\mu = 14 \text{ps/\mu m-km}$. For a typical fiber with
effective area of $50 \mu \text{m}^2$ [15], the threshold pumping power for
amplification is $23 \text{mW}$. This excellent performance is attributed to the low loss
in the fiber.

5. Conclusion

In conclusion, we showed that linear frequency dispersion together with Kerr nonlinearity can result in amplification and temporal instability of nonlinear counterpropagating waves. As the pumping increases above certain threshold, self-
oscillations are excited; upon further increase of pumping they evolve into subharmonics and chaos. With under-threshold pumping, large gain and
\begin{itemize}
\item [broad-band] amplification are to be found. The mechanism of the entire phenomenon can be explained in terms of positive distributed feedback from the nonlinear index grating formed by the two laser beams. Our calculations show that the amplifiers based on the nonlinear optical fiber pumped by counterpropagating waves with relatively low power have great potential for various applications.

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