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NONLINEAR OPTICS OF A SINGLE SLIGHTLY-RELATIVISTIC
CYCLOTRON ELECTRON

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INTRODUCTION

The interaction of microwave and optical radiation with a tiny relativistic single electron can result in strong nonlinear-optical effects\textsuperscript{1-7} based on the most fundamental mechanism of nonlinear interaction of light with matter\textsuperscript{3}. Even a slight relativistic change of mass of a single free electron may result in large nonlinear effects such as hysteresis and bistability in cyclotron resonance of the electron precessing in a dc magnetic field under the action of an EM wave\textsuperscript{1}. The relativistic mass-effect consists in the increase of the effective mass of electron, m, as its speed v, or energy W, or momentum p, increase:

\[ \frac{m}{m_0} = \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(1 + p^2/m_c^2\right)^{1/2}, \]

where \( m_0 \) is the electron rest mass, \( \gamma = W/m_0c^2 \) is the dimensionless electron energy, and \( c \) is the speed of light. Because of very low energy losses (which are due to synchrotron radiation), the relativistic change of mass to which the hysteretic resonance is attributed may be as strikingly small as \( 10^{-10} \) - \( 10^{-7} \). Consistent with predictions\textsuperscript{1}, the hysteretic (bistable) cyclotron resonance of a free electron has been subsequently observed in an experiment\textsuperscript{2} in which a single electron has been trapped in a Penning trap for a period of time as long as 10 months.

The hysteretic cyclotron resonance\textsuperscript{1-3} occurs at the main frequency, i.e., in the situation when the driving frequency, \( \Omega \), is in the close vicinity of the unperturbed cyclotron frequency, \( \Omega_0 \), see Fig. 1a (for presently available dc magnetic fields the maximum cyclotron frequency has a wavelength in the millimeter or submillimeter range). The strongly nonlinear cyclotron resonance can also be excited by optical pumping with a driving laser frequency (or frequencies) much higher than the cyclotron frequency\textsuperscript{4-7}. All these optical excitations are, in fact, multiphoton processes that can be described as relativistic nonlinear optics of a single electron. In particular, a strong cyclotron excitation can be obtained by biharmonic laser pumping when two laser frequencies, \( \omega_1 \) and \( \omega_2 \).
Fig. 1. Spectral arrangements of driving frequencies with respect to the unperturbed cyclotron frequency $\Omega_0$ for: (a) the main resonance at the frequency $\Omega \approx \Omega_0$; (b) the optical subharmonic resonance at the frequency $\omega n \approx \Omega_0$; (c) the first-order cyclo-Raman resonance at the driving frequency $\omega_1 - \omega_2 \approx \Omega_0$; (d) the arbitrary order cyclo-Raman resonance with the driving frequency $\omega_1 - \omega_2 \approx n\Omega_0$.

differ by either $\Omega$ (e.g. $\omega_1 - \omega_2 = \Omega$), see Fig. 1b, or $2\Omega$ (i.e. $\omega_1 - \omega_2 = 2\Omega$), or even $n\omega$ (i.e. $\omega_1 - \omega_2 = n\omega$), where $n$ is arbitrary integer, see Fig. 1d. These effects may be regarded as a three-photon, four-photon, and $n+2$-photon interactions respectively or, more specifically, as stimulated cyclo-Raman scattering of first, second and or $n$-th orders; for some particular propagation configurations they can exhibit the so-called isolas. It was also shown recently that a single electron can exhibit high-order cyclotron subharmonics in which the ratio of driving (single) laser frequency to the frequency of cyclotron excitation is an arbitrary integer, $n$ (this corresponds to an $n$-photon process), see Fig. 1c. The unified theory of all these effects is based on the decomposition of electron motion into a purely cyclotron component and noncyclotron components, the latter ones including higher-order oscillations with all possible frequency combinations. This theory allows one to obtain general results which are valid for an arbitrary energy of electron excitation, and for arbitrary order of interaction.

Since all of the characteristics of the nonlinear interactions of EM radiation with a single electron are directly related to fundamental constants, they can be used for measurement of these constants with greatly enhanced precision. The hysteretic excitation of a single electron can be viewed as the ultimate multistable interaction of light with matter, since it relies on the bistable interaction of an EM wave with the single simplest microscopic physical object and is based on a fundamental effect, the relativistic change of its mass. This effect also suggests optical bistability based on the intrinsic property of a microscopic object rather
than on its macroscopic property in a nonlinear medium. In the future, it could be important to study hysteresis in the vicinity of its threshold, which will provide a unique opportunity to explore the quantum limit of bistable oscillators.

In the multiphoton interaction of an optical laser with an electron, cyclo-Raman resonances offer a new method of excitation of the cyclotron motion which may prove more advantageous than conventional methods utilizing either mw or rf oscillators. Indeed, since the optical frequencies, \( \omega_1 \) and \( \omega_2 \), can be provided by two modes of the same laser, optical excitation allows for easily tunable control over the difference frequency (\( \omega_1 - \omega_2 \)). The power of laser light required to obtain cyclotron excitation is sufficiently low to allow for the use of lasers in a cw or quasi-cw regime.

The proposed effect may also be used for particle acceleration \(^4,6\); even in a simple Penning trap, kinetic energy of an excited electron as high as a few MeV may be obtained. For the \( n^{th} \) order cyclo-Raman resonance as well as the \( n^{th} \) order subharmonics, the excited electron can have \( n \) possible phases of cyclotron excitation (which differ by \( \pi/n \)); the phase which is excited depends on the initial conditions. This is a manifestation of a new type of optical bistability which may be regarded as phase multistability \(^4,6,9\) (i.e., based on multistability of the phase of oscillation rather than on multistability of its amplitude). The high-order subharmonics \(^5,10\) may provide coherent links between laser and rf or microwave frequency standards. For example, to divide the frequency of a CO\(_2\) laser (\( \lambda \approx 10\mu\text{m} \)) by a factor of 100 down to \( \lambda \approx 1\text{mm} \) in one step, cw laser as low as \( 10^{-6}\text{W} \) is sufficient \(^5\). It is worth noting that most nonlinear effects discussed above are also feasible \(^2,11,12\) in some narrow-gap semiconductors, making use of the pseudo-relativistic behavior of the effective mass of their conduction electrons \(^3,17\).

I. LORENTZ EQUATION AND RELATIVISTIC NONLINEARITIES

Consider a single electron in a homogeneous magnetic field \( H_0 \) with the unperturbed (nonrelativistic) cyclotron frequency \( \Omega_0 = e H_0/\hbar c \). The EM wave impinging on the electron may, in general, consist of any number of plane traveling waves \( \tilde{E}_j(\omega_j t - k_j \cdot r) \), where \( \omega_j \) and \( k_j \) are, respectively, the frequency and wave vector of the \( j^{th} \) traveling wave. We treat the problem classically; the motion of an electron with an arbitrary momentum \( \tilde{p} = m_e \gamma \tilde{v} \) is governed therefore by the relaxation-modified Lorentz equation \(^{17-23}\):

\[
\frac{d\tilde{p}}{dt} = e \sum_j \tilde{E}_j + \frac{e}{\gamma_0 c} \tilde{p} \cdot \tilde{\Sigma}(k_j \cdot \tilde{E}_j/k_j) + \frac{e}{\gamma_0 c} (\tilde{p} \cdot \tilde{H}_0) + e\tilde{g}_{tr} + \tilde{F}_1
\]

(1)

where \( e \) is the electron charge. The second term on the right-hand side of Eq. (1) is the Lorentz radiation force of the incident EM waves; it is attributed to the magnetic field of the EM wave, \( \tilde{H}_j = (k_j \cdot \tilde{E}_j)/k_j \). A trapping dc electric field \( \tilde{G}_{tr} = \tilde{G}_{tr}(\tilde{r}) \) is provided by the Penning trap potential.

The term \( \tilde{F}_1 \) in Eq. (1) represents energy loss of the electron. For a single electron it arises only from the so-called synchrotron radiation of the revolving cyclotron electron. In the non-relativistic case (\( |\tilde{v}| < c \)), this term can be written as \( \tilde{F}_1 = (2e^2/3c^3)\tilde{v}/dt^2 \). If the electron
oscillates in a single frequency \( \omega_0 \) (e.g., the cyclotron frequency), with all other oscillations being negligibly small, than 
\[
\frac{d^2 \gamma}{dt^2} = \omega_0^2 \gamma + \frac{\gamma}{\omega_0^2 / m_0} \text{ and the damping term can be written as } \Gamma_1 = -\Gamma_0 \hat{p}, \text{ where } \Gamma \text{ is the dimensionless bandwidth, or damping parameter, of cyclotron resonance:}
\]
\[
\Gamma = \frac{2e^2 \omega_0}{3m_0 c^3} = 2 \pi \frac{k_0}{3} \ll 1
\]
where \( r_e = e^2 / m_e c^2 = 2.8 \times 10^{-13} \text{ cm is the so-called classical electron radius} \) and \( k_0 = \omega_0 / c \). The same parameter \( \Gamma \) determines the dimensionless bandwidth of the cyclotron resonance, \( \Gamma = \Delta \omega / 2 \omega_0 \). For presently available dc magnetic fields, synchrotron radiation damping is extremely small; e.g., for \( B_0 = 6 \text{ Tesla}, \lambda_0 = 0.2 \text{ cm, and } \Gamma \approx 0.7 \times 10^{-11} \). It is precisely this fact that gives rise to the strikingly low driving threshold for observation of the non-linear effects discussed here. For example, in order to observe a hysteretic cyclotron resonance at the main frequency (see Section III below), the relativistic shift of mass, \( \Delta m / m_0 \equiv \beta^2 / 2 \) where \( \beta = v / c \), must exceed roughly speaking, the half-width of the resonance, which corresponds to very low kinetic energy. In fact, the experimental conditions (with typical kinetic energy \( 0.05 - 0.5 \text{ eV}, \) i.e. \( \beta^2 / 2 \approx 10^{-7} - 10^{-6} \)) correspond to the situation in which this threshold is exceeded by a few orders of magnitude. In a more general case in which the electron is excited to any energy, one obtains:
\[
\Gamma = -e^2 \omega_0^2 / 3c^3 (1 - \beta^2)^2 = -\Gamma_0 \hat{p}.
\]
We now introduce dimensionless momenta \( \hat{p} \), fields \( \hat{f} \), unit vector of magnetic field \( \hat{h} \) and unit wave vectors \( \hat{q} \), and trapping field \( \hat{g} \) as follows: \( \rho = p / m_0 c; \hat{f}_j = E_j / B_0 = e E_j / m_0 c \omega_0; \hat{h} = H_0 / B_0; \hat{q}_j = k_j / k_0; \hat{g} = G / \hat{r}_0 = e \hat{G} / m_0 c \omega_0 \), after which we rewrite Eq. (1) as:
\[
\dot{\hat{p}}_c = \dot{\rho}_c(t) + \rho^{(1)} + \rho^{(2)} + \ldots; |\rho^{\omega_0 / m_0}| \ll |\rho_c|.
\]
For most of the purposes discussed in this paper, in particular, for all calculations of steady-state electron excitation, the trapping potential, and therefore \( \hat{g} \), can be excluded from consideration (see below).

In order to develop a procedure that will enable us to describe all the high-order nonlinear (including multiphoton) effects, we have to distinguish pure cyclotron electron motion from all the other motions of the electron at noncyclotron frequencies. Assuming that the dc magnetic field \( H_0 \) is sufficiently strong in order for pure cyclotron motion to be dominant, we can treat all the noncyclotron motions as small perturbations. Therefore, the total momentum \( \hat{p} \) can be written in the form
\[
\hat{p} = \hat{p}_c(t) + \rho^{(1)} + \rho^{(2)} + \ldots; |\rho^{\omega_0 / m_0}| \ll |\rho_c|,
\]
where \( \rho_c \) is a "cyclotron" component of momentum describing a pure precession of the electron around some fixed center (r = 0) with the frequency \( \omega_0 \). The various orders of "non-cyclotron" components \( \rho^{(n)} \) include oscillations with all the other, non-resonant, frequencies and may have any orientation. The dynamics of the cyclotron component \( \rho_c \) are governed then by the equation:

\[134\]
\(\Omega_0^{-1}(dp_c/dt) - \gamma^{-1}_c (\rho \times \vec{h}) + \gamma^{-3}_c \rho \vec{c} = F_c^{(1)}(t) + F_c^{(2)}(t) + \ldots \) (6)

where \(\gamma_c = \sqrt{1 + \rho^2_c} \) is cyclotron motion energy. In Eq. (6) we introduced nonlinear forces \( \vec{F}^{(s)}_{\rho \rho_c} \) of different orders "s" each of which is related to the respective noncyclotron component \( \rho_{nc} \); "c" in \( \vec{F}^{(s)}_c \) labels the components that oscillate at the cyclotron frequency \( \Omega_0 \) and are normal to \( \vec{H}_0 \). Once the nonlinear force \( \vec{F}^{(s)}_c \) is determined, the respective noncyclotron component \( \rho_{nc}^{(s)} \) can be found from the equation:

\(\Omega_0^{-1}(d\rho_{nc}^{(s)}/dt) - \gamma^{-1}_c (\rho_{nc} x \vec{h}) + \gamma^{-3}_c (\rho_{nc} \rho_{nc}) (\rho \times \vec{h}) = \vec{F}^{(s)}_c \rho_{nc}^{(s)} \) (7)

In particular, the force \( \vec{F}^{(1)}_c \) responsible for the hysteric resonance at the main frequency (see below, Section II), as well as for subharmonic oscillations (see below, Section III) is defined as

\(\vec{F}^{(1)}_c = \sum_j \vec{k}_j \cdot \vec{r}_j (\omega t - k \cdot r_j) + \gamma^{-1}_c \rho \vec{c} \times \vec{f}_j \) (8)

where \( \vec{r}_c = c \gamma^{-1}_c \rho \vec{c} \partial \rho \) is the "cyclotron" radius vector. All the higher order nonlinear forces \( \vec{F}^{(s)}_c \) can be written as

\(\vec{F}^{(s)}_c = \vec{F}^{(s)}_D + \vec{F}^{(s)}_L + \vec{F}^{(s)}_R \); \((s>1)\) (9)

where each of the \( s \) th order forces \( \vec{F}^{(s)}_D \), \( \vec{F}^{(s)}_L \), and \( \vec{F}^{(s)}_R \) is defined as a sum of all terms of \( s \) th order in \( \vec{f}_j \) originating, respectively, from the first, second, and third terms on the rhs of Eq. (4) respectively, in which all the lower order terms of \( \rho \) in Eq. (5) (the highest of which is \( \rho^{(s-1)} \)) are taken into account. In particular, the force \( \vec{F}^{(2)}_c \) responsible for cycle-Raman resonances (see below, Section IV) was found in Ref. (4).

In eq. (9) we distinguish three main mechanisms of nonlinear interaction, each of which is related to the respective first three terms on the rhs of Eq. (4). The spatial oscillations of the electron make it see the phases \( k \cdot r \) of the incident fields \( \vec{f}_j (\omega t - k \cdot r_j) \) (the first term on the right-hand side of Eq. (4)) rapidly modulated since \( \vec{r} = c \gamma^{-1}_c \rho \partial \rho \) (the second term in the rhs of Eq. (4)) gives rise to the "Doppler" nonlinear mechanism, with the nonlinear force \( \vec{F}^{(2)}_D \) defined as \( \vec{F}^{(2)}_D (s) = \sum_{j} (\vec{k}_j \cdot \vec{r}_j (\omega t + k \cdot r_j (s))) \). The Lorentz radiation force \( \vec{F}^{(2)}_R \) originated by the magnetic field of incident waves (the second term in rhs of Eq. (4)) gives rise to the "Lorentz" nonlinear mechanism, with the nonlinear force \( \vec{F}^{(2)}_L = (c \gamma^{-1}_c \rho \times k \vec{f}_j (s)) \). Finally, the slight modulation of relativistic mass \( m(t) = m_0 \gamma \) results from small noncyclotron modulation of the energy term \( \gamma^{-1}_c \) in the last, cyclotron, term on rhs of Eq. (4); hence, the designation "relativistic" nonlinear mechanism, with the force \( \vec{F}^{(2)}_R \) defined as \( \vec{F}^{(2)}_R (s) = (c \gamma^{-1}_c \rho \times h) (s) \). Contributions from all these three mechanisms can be of the same order of magnitude. In general, none of them can be neglected; however, for particular propagation and polarization configurations some of them may dominate. It is worth emphasizing, however, that once the cyclotron motion is excited, it is only the relativistic mass-effect (the term \( \gamma^{-1}_c \) on the lhs side of Eq. (6)) that acts to limit the excitation energy and to form a hysteric resonance.

The dynamics of cyclotron momentum, \( \rho_c \), are described assuming
\[
\dot{\rho}_c = \rho_c (\dot{\phi} \sin(\Omega t + \phi) + \dot{\theta} \cos(\Omega t + \phi))
\]  
(10)

where \( \rho_c \) and \( \phi \) are the (unknown) slowly varying amplitude and phase of the cyclotron momentum respectively, whose dynamics are determined by Eq. (6) as:

\[
\Omega_0^{-1} \dot{\rho}_c / \rho_c + \Gamma_{\rho_c} \rho_c = \Phi, \quad \Omega_0^{-1}(d /dt) - (\gamma^{-1} - \Omega_0) = \rho_c^{-1} \Phi,
\]  
(11)

with parameters \( \Phi \) and \( \Phi \) defined as

\[
\Phi = \left( \frac{\rho_c^{\text{lowest}} \cdot \rho_c / \rho_c}{0} \right) = 0 (\Gamma); \quad \Phi = \left( \frac{\rho_c^{\text{lowest}} \cdot (\rho_c \times \dot{\rho})}{\rho_c} \right) = 0 (\Gamma),
\]  
(12)

where "0" labels zero-frequency components and \( \rho_c^{\text{lowest}} \) is a non-zero cyclotron component of the lowest order. Since \( \Gamma \ll 1 \) and \( \rho_2 \gg \Gamma \) (which is immediately attained once the threshold of excitation is exceeded), in steady state one has

\[
\gamma_0 = \Omega_0 / \Omega + 0(\Gamma), \quad \rho_c = \sqrt{(\Omega_0 / \Omega)^2 - 1} + 0(\Gamma).
\]  
(13)

I.a. Trapping Potential and Propagation Configurations

The first experimental observation of hysteretic cyclotron resonance\(^2\) at the main frequency (\( \Omega = \Omega_0 \)) predicted earlier in Ref. (1), was accomplished using an electron confined in the relatively weak quadrupole potential of a Penning trap to keep the electron from escaping. The Penning trap in Ref. (2) consists basically of a positive ring and two negative cap electrodes kept at liquid helium temperature in the core of a 60-kG superconducting magnet. The surface of the quadrupole system (the caps and the ring) is formed by a hyperboloid of revolution symmetric with respect to \( \vec{B}_0 \). In such a potential, the confined electron exhibits, in general, quite complicated behavior. Its motion may consist of the (dominant) cyclotron motion, which is the fastest one (=164 GHz in Ref. (2)), much slower magnetron motion (=60 MHz), and even much slower axial oscillations along \( \vec{B}_0 \) (=1.6 kHz). The latter oscillations were used\(^2\) to measure a relativistic mass increase through the axial frequency shift. Since we concentrate on the steady states, both magnetron and axial oscillations can be neglected. (We note though\(^1\) that both of them could form a significant component in the transient behavior of electron if either driving frequency or amplitude is swept too fast). In fact, all the steady-state hysteretic and nonlinear effects with a single electron can be obtained in a dc magnetic field without any electric static trapping potential. However, when the electron is subjected to the action of only one traveling wave, the Lorentz radiation force of this wave acting upon the electron can force it to move away from its initial position unless a trapping potential is applied. In order to avoid this complication as well as the necessity to use the trapping potential in our calculations, we will always use propagation configurations which involve only a standing wave (i.e. couples of counter-propagating traveling waves) for each driving frequency.
The type of nonlinear interaction of EM radiation with a synchrotron electron, as well as the nonlinear mechanisms involved, depends critically on the direction of propagation and polarization of driving waves with respect to the dc magnetic field \( \mathbf{H}_0 \). Although almost all of the interactions discussed here are feasible for quite a general propagation configuration, there are a few propagation arrangements each of which can be regarded as the most optimal geometry for some particular nonlinear interaction. We will use a two-letter classification of propagation configuration using the first letter to describe the wave propagation direction and the second one to describe the wave polarization with respect to \( \mathbf{H}_0 \). The "pure" excitation of hysteretic cyclotron resonance at the main frequency (with no other "side" nonlinear effects) corresponds to the "PC" propagation configuration in which driving waves propagate parallel ("P") to \( \mathbf{H}_0 \) (i.e. \( \mathbf{q}_i = \mathbf{zh} \)) and are circularly ("C") polarized in such a way that the polarization vector precesses in the same direction as the electron does. The high-order subharmonics are most efficiently excited by the "NN" propagation configuration when driving waves propagate normally ("N") to \( \mathbf{H}_0 \) and are linearly polarized in the direction also normal to \( \mathbf{H}_0 \), whereas cyclo-Raman excitations are stimulated by driving waves in the "NP" propagation configuration (i.e. by linearly polarized waves incident normally to \( \mathbf{H}_0 \)).

II. RELATIVISTICALLY-HYSTERETIC CYCLOTRON RESONANCE

The hysteretic resonance at the main cyclotron frequency, see Fig. 1a (i.e. when \( \Omega \cong \Omega_c \), where \( \Omega \) is the driving frequency and \( \Omega_c \) is the cyclotron frequency of excited electron) is perhaps the most simple and fundamental relativistic nonlinear effect based strictly on the relativistic change of electron mass\(^1\)\(^-\)\(^3\). This is a first-order effect; Eq. (8) shows that for the PC propagation configuration (see inset in Fig. 2) none of the other mechanisms are engaged. Indeed, assuming driving waves in the form \( \hat{f}(t) = (f/2) \times (\hat{\theta}_x \sin(\omega t + kq_0 \cdot \hat{r}) + \hat{\theta}_y \cos(\omega t + kq_0 \cdot \hat{r})) \) where \( j = 1, 2, kq_j = (-1)^j h, k = \Omega/c, \) and \( f = \frac{2eE/2m_0c^2\Omega_0}{(E_0^2 \Omega^2)^{1/2}} \) (with \( |E|^2 \Omega \) being the intensity of each of the waves), and assuming that the center of the electron orbit is located at the origin, \( \hat{r} = 0 \), we obtain \( \gamma_0^0 = \gamma_0^0(t) = f_1^2 + f_2^2 = \gamma_c \sin \omega t + \gamma_y \cos \omega t \). The dynamics of \( \rho_c \) and \( \phi \) are governed by Eqs. (11) with \( \Phi = f \cos \phi, \) \( \phi = -f \sin \phi \). The steady-state solution (\( d/dt = 0 \)) is thus determined by

\[ f^2 = \rho_c^2 (\gamma_0^0 - \gamma_0^0) ; \tan \phi = -(\gamma_c \Omega/\Omega_0 - 1)/\gamma_0^2. \]

(14)

Since \( \Gamma \ll 1 \), the hysteretic (bistable) resonance can be achieved at very low excitation, i.e., \( \rho_c \ll \beta_c \ll 1, \gamma_c \approx 1 + \beta_c^2/2 \), in which case Eq. (14) reduces to

\[ f^2 = \beta_c^2 (\Delta^2 + (\Delta + \beta_c^2/2)^2), \]

(15)

where \( \Delta = \Omega/\Omega_0 - 1 \ll 1 \) is a dimensionless resonant detuning parameter, see Fig. 2.

Under threshold conditions:
Fig. 2. The plot of normalized kinetic energy of the electron $\beta_c^2/2$ vs normalized resonant detuning $\Delta/T$ for various driving amplitudes of incident EM field. Curves: (1) $f < f_{th}$, (2) $f = f_{th}$, (3) $f > f_{th}$. The inset depicts the propagation configuration of driving radiation.

$$f^2 > f_{th}^2 \equiv (16/3\sqrt{3})T^3, \Delta < \Delta_{th} \equiv -r\sqrt{3},$$  \hspace{1cm} (16)

Eq. (15) yields a three-valued solution for $\beta_c$ with two of the values stable solution and one an unstable solution (Fig. 2). At the threshold point the excitation is $\beta_{th}^2 = 2T\sqrt{3}$ (curve 2 in Fig. 2), and the orbit radius is $r = \beta_{th} / k \ll \lambda$.

For the experimental conditions, $\gamma \approx 0.66 \times 10^{-11}$, and Eq. (16) yields $E_{th} = 2.65 \times 10^{-10}$ V/cm which corresponds to a classical threshold intensity as strikingly low as $\approx 10^{-18}$ W/cm$^2$. The classical estimate (16) in electron which is nearly $\alpha^{-1}$ times smaller than a quantum limit $\hbar \Omega (\alpha = e^2 / \hbar c = 1/137$ is the fine-structure constant). Therefore, in the close vicinity of the threshold (16), only the quantum approach can give an adequate description of the phenomenon, whereas, for a sufficiently strong driving field ($E^2 \gg E_{th}^2$) the classical results (in particularly, hysteretic jumps) remain valid. The actual excitation used in the experiment was a few orders of magnitude higher than this threshold, i.e., the experimental data obviously represent the classical limit. The further improvement of the stability and bandwidth of the source of the driving radiation may hopeful bring the experiment to the threshold level and, therefore, to the quantum limit of the hysteresis effect.

In the case of the three-valued solution, the examination of Eqs. (11) linearized in the close vicinity of the steady state solutions, Eqs. (14), shows that only those states are stable which satisfy the energy criterion $d(\beta_c^2)/df^2 > 0$ (solid branches of the curves in Fig. 2); otherwise, they are unstable (dashed branches in Fig. 2). The smallest of the hysteretic jumps, the up-jump from the lower excitation branch to the higher one, occurs in the close vicinity of the unperturbed resonance, whereas the much stronger down-jump may occur considerably far from that point (curve 3 in Fig. 2). When the driving radiation is sufficiently strong but the motion
is still low-relativistic (i.e., \( \beta^2 \gg \beta^2_{th} \), but \( \beta^2_{c} \ll 1 \)) the kinetic energy of the excited electron \( \gamma = 1 = \beta_{c}^2/2 \) at the upper branch of hysteretic curve depends almost linearly on the detuning \( \Delta \):

\[
\beta_{c}^2/2 \approx -\Delta. \tag{17}
\]

The experimental data \(^2\) are in good agreement with Eq. (17); according to them, \( |2\Delta/\beta^2| \approx 0.92 \) which is quite close to the value \( 1.0 \) suggested by Eq. (17). This confirms that the nonlinearity of the system is attributable to the relativistic mass-effect. In terms of nonlinear oscillation theory, the hysteresis in a nonlinear oscillator based on a low-relativistic electron resembles the so-called vibration hysteresis in driven anharmonic oscillators\(^2\) (in particular, the so-called Duffing oscillator, e.g., the pendulum) or in nonlinear parametric systems\(^2\).

In principle, it is feasible to obtain electron with quite a relativistic energy in a standing wave configuration even with modest driving power. For example, in order to excite an electron up to \( \gamma_{c} = 188 \) (i.e., to the energy \( = 95\text{MeV} \) with orbit diameter \( \approx 6\text{cm} \)) with \( H_0 = 100\text{kG} \) (\( \lambda \approx 1\text{mm} \)) one needs \( q \) driving source at \( \lambda \approx 18.8\text{cm} \) with an intensity still as low as \( 0.07\text{W/cm}^2 \).

III. HIGH-ORDER OPTICAL SUBHARMONICS

The high order subharmonics with \( \omega = n\Omega \), see Fig. 1b, in nonlinear cyclotron resonance give a very interesting example of a multiphoton, highly nonlinear process which is still due to the first-order nonlinear force \( F^{(1)} \), Eq. (8). The most efficient generation of these subharmonics occurs with the NN propagation configuration, in which case the Doppler and Lorentz nonlinear mechanisms contribute to the nonlinear interaction. The electron is driven by two counter-propagating plane waves \( \tilde{E}_j = \tilde{E}_j f \cos(\omega t + (\pm 1)^j (k x - \psi)) \); \( j = 1,2 \); \( k = \omega/c \) (with the same frequency \( \omega \) and amplitude \( f \)) both of them propagating along the axis x normal to \( H_0 \), polarized along the axis y, also normal to \( H_0 \) (see inset in Fig. 4), and forming a standing wave pattern. When the subharmonic of the \( n^{th} \) order is excited, the momentum of electron is described by Eq. (10), where \( \Omega = \omega/n \).

In our further calculations, we assume that the center of the orbit is located in the center of a trap and coincides with a maximum of the total field \( \tilde{E}_1 + \tilde{E}_2 \) (which corresponds to \( \psi = 0 \)) when \( n \) is odd, and with a node of the total field (i.e. \( \psi = \pi/2 \)) when \( n \) is even. Separating the cyclotron component \( \tilde{E}^{(1)}_{c} \) out of \( \tilde{E}^{(1)} \), and using Eqs. (12) one finds that in the steady state \( (\partial/\partial t = 0) \) the energy of the excitation is

\[
\gamma_{c} = n\Omega/\omega + \Delta\gamma_{c} \quad \text{with} \quad \Delta\gamma_{c} = 0(\Gamma) \ll 1. \quad \text{The small detuning} \quad \Delta\gamma_{c} \quad \text{for} \quad n \geq 2 \quad \text{is found to be}
\]

\[
\frac{\Delta\gamma_{c}}{\Gamma} = \pm \frac{2J_{n}(n\beta_{c})}{\beta_{c}^{2}Y_{c}} \sqrt{\frac{2\gamma_{c}}{\beta_{c}^{2}Y_{c}}} \sqrt{\frac{4}{(J_{n-1}(\beta_{c}) - J_{n+1}(\beta_{c}))^{2}}}
\tag{18}
\]

\[\gamma_{c} \]

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Fig. 3. Kinetic energy of the cyclotron motion $\gamma - 1$ vs. the dimensionless frequency of a driving field $\omega/\Omega_0$ for various orders of subharmonics $n$ ($n = 1$ is the main resonance, $n = 2$ is second subharmonic, etc.) and for the fixed amplitude parameter $\mu = f/\Gamma = E/E_0$ of a driving field. (a) $\mu = 0.9$, (b) $\mu = 3.5$, (c) $\mu = 16$. The thick solid branches correspond to stable states, and thick broken branches to unstable states. All the branches are stretched out along lines determined by the formula $\gamma = n\Omega_0/\omega$ for each order $n$.

where $J_\nu$ is an ordinary Bessel function of the $\nu$th order and $\mu = f/\Gamma = 3E_0^2/2e\Omega_0^2$ is a driving parameter. The frequency of subharmonic $\Omega = \omega/n$ follows closely the effective cyclotron frequency $\Omega_c = \Omega_0/\gamma_c$ determined by the relativistic mass-effect $m/m_0 = \gamma_c$, see Fig. 3.

A stability analysis using Eq. (11) shows that the positive upper sign in Eq. (18) corresponds to stable states, and the negative sign to unstable states. For each stable value of the energy $\gamma_c$, the cyclotron motion can have $n$ equally possible different equidistant states of the phase $\phi = \phi(n, f, \omega) + 2\pi s/n$, $s$ being integral, $0 \leq s < n$. This property is common for any subharmonic oscillations of the $n$th order regardless of their origin$^{22,23}$.

For any order $n > 2$, there are both upper and lower limits on the energy of excited subharmonics$^5$ for any fixed driving amplitude $f$ (or $\mu$), see Figs. 3 and 4, which give rise to the formation of isolated branches of excitation for each individual subharmonic with $n > 2$. The initial "jump-start" required for the electron to reach the desirable subharmonic, can be provided either by injecting electrons with proper energy, or by triggering the system with the microwave oscillator having its cyclotron frequency as in Section II (or some of its higher harmonics) near the cyclotron frequency, or with optical biharmonic pumping$^{3,6}$ by a laser having $\omega_1 - \omega_2 = \Omega$, see Section IV below. For any $n \geq 2$, there is a minimum (threshold) driving amplitude $\mu_{th}$ which is an increasing function of $n$ with $\mu_{th}(n = 2) = 1$. The threshold amplitude
Fig. 4. Extremum kinetic energy $\gamma - 1$ of the cyclotron motion vs. driving parameter $\mu$ for various orders of subharmonics $n$. Inset: incident light configuration with respect to the dc magnetic field $H_0$.

$\mu_{th}(n)$ corresponds to some certain energy of excitation which increases (although very slowly) as $n$ increases.

The intensity of driving laser radiation required to excite even very high-order subharmonics, is very low. Indeed, the threshold amplitude for excitation 2-nd order subharmonic ($\mu_{th} = 1$) is

$$E = \frac{2eU_0^2}{3c^2} = \frac{(2/3)(m_e c^2/e)k_0^2}{m_e r}$$ (19)

For $\lambda \approx 1$ mm, $E_0$ corresponds to a low intensity as $\approx 2 \cdot 10^{-10}$ W/cm$^2$. Consider now an example when the laser wavelength is $\lambda = 10$ $\mu$m (CO$_2$ laser), whereas $\lambda = 1$ mm, and therefore $n = 100$, i.e., the laser frequency is divided by a factor 100 in one step. The calculations give $\mu_{th} \approx 560$ such that even if $\mu \approx 4\mu_{th}$ the resulting intensity is still very low, $\approx 10^{-7}$ W/cm$^2$. With the area of the beam $\lambda \times \lambda \approx 10^{-4}$ cm$^2$ this translates into a total driving power as low as $10^{-6}$ W.

High-order subharmonics have been observed in many resonant nonlinear systems (both mechanical and electrical ones). For example, in a simple nonlinear circuit using a biased diode as a nonlinear capacitor, a one-step frequency division by a factor up to 500-1000 in the ultrahigh rf-range has been observed and studied$^{23}$. The subharmonics in ref. (23) were attributable to the self-synchronization of parametric oscillations induced by a driving force. The same principle has later been proposed$^{24}$ to obtain low-order subharmonics in the optical range using an optical parametric oscillator. The subharmonic excitation of cyclotron motion by radio-frequency or microwave driving sources is well known; in fact, the synchrotron$^{25}$ and synchro-cyclotron$^{26}$ principles of particle acceleration are based on driving a particle beam at a frequency equal to the multiplied cycling frequency of accelerated particles. A similar principle was recently proposed$^{10}$ to obtain mw subharmonic radiation of electrons using a laser as a driving source.
In both mw accelerators and the laser schemes just described, the important condition is that the driving field must be highly inhomogeneous in space and act upon particles over a distance much shorter than the orbit circumference. On the other hand, the subharmonic described here can be obtained with homogeneous plane standing (or traveling) waves acting upon a particle along its entire cyclotron orbit. The laser power required for such a subharmonic excitation is so low that virtually any cw infrared laser can be used for this purpose. The quest for optical one-step multiple transformation of frequency (by either multiplication or division) stems from the need to cover a gap between optical and microwave time and frequency standards. The conventional techniques are based on frequency multipliers 27, complex frequency-synthesis chains 28, frequency division based on locking both a laser and a rf source to a cavity 29, etc. The high-order cyclotron subharmonics discussed here have potential to provide a promising alternative method for obtaining a direct coherent link between lasers and microwave frequency standards.

IV. CYCLO-RAMAN OPTICAL RESONANCES

Electron excitation at the cyclotron frequency \( \omega_c \) by an optical biharmonic laser with two frequencies \( \omega_1 \) and \( \omega_2 \) (\( \omega_1 > \omega_2 \)) such that each one of them is much higher than \( \Omega \) and their difference equals to \( \Omega = \Omega_c \), \( \omega_1 - \omega_2 = n \Omega \) (see Fig. 1c) is essentially a three-photon process that can naturally be regarded as stimulated cyclo-Raman scattering (of the lowest order). More general stimulated cyclo-Raman scattering of arbitrary order \( n \) is feasible with \( \omega_1 - \omega_2 = n \Omega \) (see Fig. 1d). The cyclo-Raman scattering with lowest orders \( n = 1, 2 \) was considered in Ref. (4) for the PC propagation configuration at low excitation energy, and in Ref. (6) for the NP propagation configuration and \( n = 1 \) at arbitrary excitation energy. For any order \( n > 2 \), the cyclo-Raman resonances in the NP configuration exhibit "prohibited" and "allowed" cyclotron orbits (which result in multiple isolated branches of solutions, the so-called isolas) \( n \) possible equidistant phase state (which results in phase multistability for \( n > 1 \)), the optical Stark-shift (i.e., an intensity-dependent shift of the eigen-frequency), and multi-wave mixing effects.

We assume here that neither ratio \( (\omega_1 + \omega_2)/\Omega \) nor \( \omega_1/\Omega \) is integer (i.e., higher-order subharmonics, Section III, are excluded by proper frequency tuning). Because of such a choice, \( F^{(1)}(1) = 0 \) in Eq. (6). Therefore, cyclo-Raman scattering is attributed to the second-order nonlinear force \( F^{(2)} \). We also choose the NP configuration in which all the optical traveling waves \( \vec{f}_j \) propagate in the plane normal to \( \vec{h}_0 \) with their polarization parallel to \( \vec{h}_0 \) (see inset in Fig. 5). For this configuration \( F^{(2)} \) is reduced to

\[
F^{(2)} = \gamma_c^{-1} \sum_j q_j \vec{f}_j^{(1)} \vec{f}_j^{(1)} \rho_{\text{nc}} - \frac{\rho_{\text{nc}} \times \vec{h}}{2 \gamma_c^3} (\rho_{\text{nc}})^2 \tag{20}
\]

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where the first noncyclotron component of momentum, \( \rho_{nc} \), Eq. (7), is now \( \rho_{nc}^{(1)} = \Omega_0/\Omega_0 \) \( \rho_{nc} \), given by \( \dot{\rho}_{nc}^{(1)} = \Omega_0/\omega_0 \)

\[
\dot{\rho}_{nc}^{(1)} = \Omega_0/\Omega_0 \int_0^\infty f_0^{(1)} \, \frac{\partial}{\partial \xi}(\Omega_0/\omega_0) \, f_j(\omega_j t - k_j \cdot r_0 - \pi/2).
\]  

(21)

If the driving radiation at both frequencies \( \omega_1 \) and \( \omega_2 \) forms standing wave patterns with all the waves propagating along the same axis \( x \), i.e., \( \vec{\eta}_{i\pm} = \vec{\delta}_{i\pm} \), the electric fields of these standing waves are expressed as \( \vec{E}_{1\pm} = \vec{f}_{1\pm} + \vec{f}_{1\mp} \) \( (i = 1, 2) \) with \( \vec{f}_{1\pm} = f_{1\pm} \sin(\omega_1 t \pm k_1 x + \psi_{1\pm})/2; i = 1, 2 \), where \( f_{1\pm} \) are counter-propagating traveling waves with the same amplitude \( f_{1\pm} \) and frequency \( \omega_i \); their phases \( \psi_{1\pm} \) in general could be different.

Steady-state cyclotron motion is achieved when the center of the cyclotron orbit coincides with a node of one of the standing waves and simultaneously with the maximum of the other one if \( n \) is odd, or when the center of the cyclotron orbit coincides with either nodes or maxima of both standing waves if \( n \) is even. At such a point, the average radiation forces of all the waves acting upon the electron cancel each other; one of the choices for the phases is then \( \psi_{1+} = \pi n, \psi_{1-} = 0, \) and \( \psi_{2+} = 0 \).

In the steady-state regime the small detuning \( \gamma_c / \Omega_0 - 1 \) is found to be \( \gamma_c / \Omega_0 - 1 = S + \rho_c^{-2} (Q_+ - Q_-)/(\alpha_1 + \rho_c^{-2} Q_+ - \rho_c^{-2} Q_-) \)

(22)

where \( S \) is the so-called optical (i.e. intensity-dependent) Stark-shift of the relativistic cyclotron frequency \( \Omega_c \), and \( Q_\pm \) are defined as \( Q_\pm = \sum_{n=1}^\infty (\alpha_n + \rho_c^{-2} \alpha_n) + \rho_c^{-2} \alpha_n \). In these equations, \( \alpha_n \) is the index of the \( n \)-th order Bessel function of first kind, \( n \), \( \alpha_n \) is the index of the \( n \)-th order Bessel function of second kind, \( n \), \( \alpha_n \) is the index of the \( n \)-th order Bessel function of first kind, \( n \), \( \alpha_n \) is the index of the \( n \)-th order Bessel function of second kind, \( n \), and \( \alpha_n \) is the index of the \( n \)-th order Bessel function of first kind, \( n \). Stability analysis of the dynamic equations shows that the upper sign in Eq. (22) corresponds to stable states, and the lower to unstable ones. In Fig. 6, stable branches are shown by solid lines and unstable ones by broken lines. It is interesting to note that the \( n \)-th order cyclo-Raman resonances (with \( n > 1 \)) exhibit phase multistability essentially similar to phase multistability in the \( n \)-th order subharmonics (Section III).

Consider first the lowest order resonance \( n = 1 \) with sufficiently low excitation energy (i.e. \( \alpha_1, \alpha_2 << 1 \) and therefore \( \rho_c = \rho_c \ll \Omega_0/\omega_1, \omega_2 \)). In such a case, Eq. (22) is reduced to Eq. (15), where instead of amplitude \( f \) of the resonant driving wave, one has to substitute now an "effective" amplitude of a three-photon interaction \( \mu \). Therefore, in order to obtain hysteresis, curve 1 in Fig. 6, one has to have \( \mu \) exceed a threshold \( \mu_{cr} = 1.75 \mu^3/2 \). In the example with \( \lambda_0 = 2 \) mm and \( \lambda_{1/2} = 2 \pi \kappa_{1/2} = 0.69 \) \( \mu \) (He-Ne laser), with \( f_1 = f_2 \), we obtain a critical amplitude, \( \mu_{cr} = 6 \) \( \mu \), which corresponds to a 48 \( \mu \)W/cm² intensity. If the beam is focused to a spot of 2 \( \mu \) diameter, this amounts to a total power of only 1.5 x 10^{-6} W.

When the driving intensity and, therefore, the excitation electron energy increases, a new feature appears, which consists of the formation
Fig. 5. The maximal and minimal values of excitation characteristic \( \alpha \) vs. the intensity of driving waves \( I \). Curve 1 (broken line with dots) and the lower branch of curves 2 (solid lines) correspond to the maximum of the main hysteresis for the cyclo-Raman scattering with \( n = 1 \) and \( n = 2 \) respectively; next above the lower branch of curves 2 corresponds to the first isola, etc. Curve 3 (broken line) corresponds to the maximal and minimal excitation of the first isola for \( n = 3 \). Areas surrounded by each curve correspond to allowed excitation; areas between them, to prohibited excitation. Curve 2' (dots) corresponds to the maximal excitation for a different configuration of the cyclo-Raman scattering with \( n = 2 \). The inset depicts the propagation configuration.

of isolated branches of excitation (the so-called "isolas"\(^6\) known also in other areas of nonlinear physics\(^7\)). This feature is peculiar to the configuration in which the waves propagate normally to the dc magnetic field and therefore form a spatially oscillating pattern in the plane of cyclotron motion. The isolas in consideration can be obtained even at low excitation, \( p_c^2 \ll 1 \) (but with sufficiently large parameter \( \alpha \)). When \( \mu \) exceeds some level, there are ranges of momentum \( p_c \) (such that \( p_c^2 \ll (p_c^2)_{\text{max}} \)), in which the steady-state excitation does not exist\(^6\), i.e., some orbits are "prohibited". For sufficiently large \( \alpha \), one obtains as the radii of permitted orbits, \( r_{\text{proh}} = (2l + 1) \lambda / 8 \), where \( l \) is an integer and \( \lambda = 4\pi c/(\omega_1 + \omega_2) \), see Fig. 5. Prohibited orbits correspond to the destructive interaction of both of the waves with respect to the electron, as opposed to the constructive interaction leading to "allowed" orbits. As the intensity of driving waves increases, the first isola is formed, then the second, and so on. The formation of isolas in the case \( n = 1 \) is illustrated in Fig. 6, in which the excitation characteristic \( \alpha \) is depicted vs. the frequency detuning parameter \( \xi = \text{sign}(\Delta) \sqrt{2 \left| \Delta / \Delta_0 \right| (\omega_1 + \omega_2) / \Delta_0} \) (\( \Delta = \Delta_0 - 1 \), and "sign" is the sign function) for various driving amplitudes \( \mu \). The first isola (see curve 3 in Fig. 6) appears at \( \alpha = 5 \); CO\(_2\) laser intensity as low as \( 77 \text{W/cm}^2 \) (\( \lambda_0 \approx 10 \mu\text{m} \), \( \lambda_{1,2} \approx 10 \mu\text{m} \), \( f_1 = f_2 \)) is required to observe it. Fig. 6 shows also a curious feature of these regimes, the self-crossing of steady state amplitude that occurs both in the isolas and in the main "mother" curve. It is unlikely, though, that this feature can be seen experimentally, since one of the
Fig. 6. The first-order cyclo-Raman (n = 1) excitation characteristic \( \alpha \) vs. its frequency detuning parameter \( \xi \) (see explanation in the text) for the fixed driving parameter \( \mu \).

Curves:
1 - \( \mu < \mu_{s.c.} \) (\( \mu_{s.c.} \) is a critical magnitude of \( \mu \) for the first self-crossing to occur);
2 - \( \mu = 7\mu_{s.c.} \) (the first self-crossing appears in the main "mother" curve);
3 - \( \mu = 2.6\mu_{c.r.} \) (\( \mu_{c.r.} \) - a critical magnitude of \( \mu \) for the first isola formation);
4 - \( \mu = 4\mu_{c.r.} \), the formation of the second isola.

The solid branches in curves 1-4 correspond to stable states, the broken ones to unstable states.

self-crossing branches is unstable. Resonances with \( n \leq 3 \) do not have a hysteretic branch; only isolas excitation is possible, although the critical laser intensity remains relatively low (e.g., an intensity \( \approx 9.7 \text{W/cm}^2 \) is needed to observe the first isola for \( n = 3 \)).

The total power \( P_C \) of the synchrotron radiation at the cyclotron frequency \( \Omega_C \) in the low-relativistic case (\( \rho_C^* \ll 1 \)) can be written as \( P_C = \eta \rho_C^* \Omega_C^2 \). One can see from Eq. (21) that due to Doppler phase modulation, the noncyclotron momentum \( \rho_C^* \) (parallel to \( \hat{H}_0 \)) oscillates at combination frequencies \( \frac{h}{2} \pm l\Omega \), where \( l \) is integer, and therefore, gives rise to stimulated dipole radiation at these frequencies with its polarization parallel to \( \hat{H}_0 \). In nonlinear optics, such a process is sometimes referred to as multi-wave mixing whereby several waves with different frequencies and wave-vectors are coupled via the nonlinear interaction. At low excitation the power \( P_2 \) absorbed from the higher laser frequency, \( \omega_1 \), and the power \( P_1 \) radiated at the lower frequency, \( \omega_2 \), obey the Manley-Rowe relationships, \( P_1 = -P_C \omega_1/n\Omega \), \( P_2 = P_C \omega_2/n\Omega \), which reflect a quantum balance of optical emission and absorption in the system and suggest a stimulated emission at the frequency \( \omega_2 \) analogous to stimulated Raman scattering. The radiation at multi-wave optical frequencies \( \omega_1, \omega_2 \pm l\Omega \) and at the mw cyclotron frequency \( \Omega \) may provide an experimental method for the observation of cyclo-Raman scattering of any order.
CONCLUSION

We have discussed the nonlinear interaction of EM (in particular, optical) radiation with a single cyclotron electron, and demonstrated the feasibility of many strongly nonlinear effects. They include hysteresis and bistability at the main cyclotron resonance and various multi-photon processes (in particular, subharmonics and stimulated cyclo-Raman scattering of arbitrary order) which exhibit isolas, prohibited and allowed orbits, phase multistability and Stark shift. All of these effects are based on the most fundamental relativistic properties of both electron and radiation such as excitation-dependent mass-effect, the Doppler effect, and the Lorentz radiation force. Most of these effects can be obtained using the cw regime of any conventional laser. Some of these nonlinear effects have the potential for applications (e.g. cyclotron mw excitation of electrons by lasers via cyclo-Raman resonance, coherent links between the mw, millimeter, and optical ranges via high-order subharmonics, etc.). Further research should investigate the "non-ensemble" quantum theory of the phenomenon.

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