Compton light pressure and spectral imprint of relic radiation on cosmic electrons

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A fully relativistic/QED theory of particle motion damping by light pressure of an isotropic, in particular relic, radiation yields analytic results for the momentum decay of an electron over entire span of energies in the Universe. Using a Fokker-Planck-like equation, the evolution of initial extremely-high-$T$ electron distribution to low-energy spectral line is analytically obtained for momentum span over tens orders of magnitude. A strong imprint of low-$T$ relic radiation on high-energy spectral distribution transpires via formation of “frozen non-equilibrium” state of electrons in current epoch, and possible existence of narrow spectral lines as remnants of past high-$T$ sources.

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Beginning with its discovery [1], light pressure played an expanding role in electrodynamics, optics, astrophysics, and dynamics of micro- and macro objects: comet tails [1c], laser levitation [2] and mirrors dynamics [3], solar sail [4], light lift [5], cooling of atoms by red-shifted laser [6] and their coherent motional excitation by blue-shifted one [7], etc. A related field-gradient, or ponderomotive force [8] has also many manifestations and applications: Kapitza-Dirac effect [9], laser trapping of atoms [5] and macro-particles [10], high-field ionization of atoms [11], etc. Relativistic effects dramatically affect that force by reversing its sign in a strong laser field [12].

In application to cosmology, of the most interest is a light pressure by an isotropic Cosmic Microwave Background (CMB, or relic) [13] radiation on hot charged light pressure by an isotropic Cosmic Microwave Back- that force by reversing its sign in a strong laser field [12].

Beginning with its discovery [1], light pressure played an expanding role in electrodynamics, optics, astrophysics, and dynamics of micro- and macro objects: comet tails [1c], laser levitation [2] and mirrors dynamics [3], solar sail [4], light lift [5], cooling of atoms by red-shifted laser [6] and their coherent motional excitation by blue-shifted one [7], etc. A related field-gradient, or ponderomotive force [8] has also many manifestations and applications: Kapitza-Dirac effect [9], laser trapping of atoms [5] and macro-particles [10], high-field ionization of atoms [11], etc. Relativistic effects dramatically affect that force by reversing its sign in a strong laser field [12].

In application to cosmology, of the most interest is a light pressure by an isotropic Cosmic Microwave Background (CMB, or relic) [13] radiation on hot charged particles resulting in the loss of their momentum/energy [14]. For high-energy baryons fast decay is facilitated by the pion production via so called $\Delta$-resonance of a baryon and high-energy photons [15] via secondary production of virtual electron-positron pairs, resulting in the existence [16] of an upper limit of the energy of cosmic rays ($\sim 0.5 \times 10^{20} eV$). In the case of leptons (e. g. electrons), due to their fundamental nature, the decay can be treated more thoroughly using QED-related dependence of photon scattering cross-section on photon energy due to the virtual pair creation and annihilation described analytically by the Klein-Nishina theory [17].

From QM viewpoint, the radiation pressure investigated beginning with work [18], is a result of averaging over ensemble of Inverse Compton Scattering (ICS) events [19], whereby a particle transfers part of its momentum to a scattered photon. The known results provide essentially asymptotic descriptions of the process, with quantitative results known for either low-relativistic “cold” electrons, or very “hot” ones in a QED domain. In this Letter, by deriving a general light pressure formula in the entire span of energies in the Universe, integrating it numerically and finding amazingly precise analytic interpolation in that domain, we found an analytic solution for the decay from extremely high to low momenta, as well as for the transformation of initial thermal electron spectra into low-energy narrow lines. It predicts the formation of a “frozen non-equilibrium” state of electrons, and existence of very narrow spectral lines as the remnants of initial high-$T$ distributions due to “pile-up” effect, both of which manifest a strong “imprint” of CMB radiation on current distribution of electrons. Our theory here is limited only to the momentum decay due to CMB radiation, and does not consider other evolution channels (such as e. g. synchrotron radiation due to galactic magnetic fields, secondary effects due to decay of protons, anisotropy fluctuations due to Sachs-Wolfe effect, etc.).

We assume a radiation with a certain spectrum $\rho_\gamma(\omega)$, isotropic and homogeneous in a preferred $L$-frame, with $N_L = \int_0^\infty \rho_\gamma(\omega) d\omega$ being the total number of photons per $L$-frame unity space. A particle moves in the $x$-axis in the $L$-frame with velocity $\vec{v} = v_\omega \hat{e}_x$, and is at rest in a $P$-frame. A light pressure $\vec{F} = d\vec{p}'/dt = F \hat{e}_\omega$ on a particle is then nonzero if $v \neq 0$; here $\vec{p}' = \vec{p}_\omega$ is its momentum in the $L$-frame, $p/m_0c = \gamma = \sqrt{\gamma^2 - 1}$, where $\beta = v/c, m_0$ is a particle rest mass, $c$ = speed of light, and $\gamma = 1/\sqrt{1 - \beta^2} = \sqrt{1 + \mu^2}$. In [20], Eq. (16), we derived a fully relativistic formula for the force $F$ based on Lorentz transformation of $\rho(\vec{k})$ from the $L$-frame to $P$-frame; it is valid for any frequency dependence of a full cross-section $\sigma(\omega)$ of scattering of an $\omega$-photon at a particle [21]:

$$F(\mu) = h \int_0^\infty \frac{\omega^2 \sigma_{ST} (\omega)}{2\gamma^2} \left\{ \int_{-1}^{1} \frac{\rho_\gamma (\omega \gamma (1 + \beta \zeta))}{(1 + \beta \zeta)^2} d\zeta \right\} \omega d\omega \tag{1}$$

Here $\sigma_{ST}(\omega) = \sigma(\omega) - \sigma_\gamma(\omega)$ is the cross-section of momentum transfer from an $\omega$-photon to a particle with a correction $\sigma_\gamma (\ll \sigma)$ to $\sigma$ [20] which is due to selecting only $x$-axis projections of scattered photon momenta. In a particular case of a free electron, their spectral profiles are depicted in [20], Fig. 1. Eq. (1) fully incorporates ensemble averaging over all the relevant parameters, which makes pressure $F$ the best tool to explore electron de-acceleration in EM field. Black-body spectral, $\rho_{BB}$, and total energy, $W_{BB} = h \int_0^\infty \omega \rho_{BB}(\omega) d\omega$ densities in the $L$-frame at the temperature $T \approx 2.725 K$ for CMB are

$$\rho_{BB}(\omega) d\omega = \frac{\omega^2}{\pi^2 c^3} e^{\omega/k_B T} - 1, \quad W_{BB} = \frac{8R^4}{15} W_c \theta^4 \tag{2}$$

for $R = 15,000$ in the $L$-frame, with $\theta = 0.024$;
which is a familiar Planck density, where $k_B$ is the Boltzmann constant, $W_c = m_e c^2 / \lambda^3$ is a “Compton energy density”, $\lambda_c = 2\pi h / m_0 c$ is the Compton wavelength, and $\theta = k_B T / m_0 c^2$ is the Compton temperature. With $\rho_L = \rho_{BB}$, Eq. (1) is reduced to a single integral over $\omega$ by using the dilogarithm function or related Bose-Einstein integrals. For $\theta \ll 1$, the entire energy span can be broken into two distinct domains in the energy space in the P-frame: (a) a classical, “Thompson domain”, characterized by a constant cross-section (see e.g. [22]) $\sigma = \sigma_0 = (8\pi/3)^2\gamma$, where $r_0 = c^2 / m_0 c^2$ is a classical EM-radius of electron, whereby the photons in the P-frame are still “cold”, $h\omega / m_0 c^2 \ll 1$, hence $\mu < O(\theta^{-1})$ (yet, since $\theta \ll 1$, it might be still strongly relativistic, $\mu \propto \gamma \gg 1$), and (b) QED ultra-relativistic “Compton domain”, $\mu > O(\theta^{-1})$. Here, due to creation & annihilation of electron-positron pairs, the full cross-section, $\sigma = \sigma_{\text{SN}} - \sigma_c$ decreases as photon energy $\epsilon = h\omega / m_0 c^2$ increases. This is described in the first order of $\alpha = e^2 / hc \approx 1/137$ by the well-known Klein-Nishina [17] formula for $\sigma = \sigma_{\text{SN}}$, see [20], Eq. (20), and $\sigma_0 \ll \sigma_{\text{SN}}$, Eq. (22); $\sigma_0 = 0$ at $\epsilon = 0$. In the limits of “cold” and “hot” radiation we have respectively $\sigma_{\text{SN}} / \sigma_0 \approx 1 - 2\epsilon$, $\sigma_0 / \sigma_0 \approx 6\epsilon/5$ at $\epsilon \ll 1$, and $\sigma_{\text{SN}} / \sigma_0 \approx (3/8\epsilon) [\ln(2\epsilon) + 1/2]$, $\sigma_0 / \sigma_0 \approx (1/\epsilon^2) [1 - 3 \ln(1 + 2\epsilon) / 4\epsilon]$ at $\epsilon \gg 1$, see [20], Fig. 1. In the Thompson domain Eq. (1) yields

$$F \equiv dp/dt = -(4/3)\sigma_0 W_L \mu r_0, \quad \text{with} \quad \gamma = \sqrt{1 + \mu^2}$$

(3)

where $W_L$ is energy density in the L-frame ($W_L = W_{BB}$ for black body). At $\mu \ll 1$, consistent with known results [22] we have $- F \ll \mu$, while in a relativistic case, $\mu \gg 1$, we have $- F \ll \mu / \mu_0$, which is reminiscent of a drag force in liquids and gases: linear in velocity (or $\mu$ here) for laminar flow (Stokes force), and $\mu / \mu_0$ for turbulent flow.

Eq (3) readily solved for $\mu(t)$ for the entire domain; with an initial condition $\mu = \mu_0$ at $t = 0$, we have

$$\mu(t) = \frac{1}{\sinh(t/t_{BB} + S_0)}; \quad t_{BB} = \frac{3\mu_0 c^2}{4\sigma_0 W_{BB}} = \frac{t_c}{\theta}$$

(4)

where $S_0 = \ln[(1 + \gamma_0) / \mu_0]$, and $t_c = 135\lambda_c / 64\pi^4 c^2 \approx 3.25 \times 10^{-18}\text{s}$ is a “Compton time”; $t_c \propto h^3$. Using it, we introduce a dimensionless time $\tau = t / t_{BB}$, and force $f = F(t_c / \mu_0)$. (It is worth noting that a “U-scale” $t_{\text{SN}} = (t_c / t_0)^{1/4} \approx 1.65 \times 10^{-9}$ where $t_0 = 4.4 \times 10^{17}\text{s}$ is the age of the Universe, comes close to the current CMB temperature, $\theta \approx 0.543 \times 10^{-9}$.) At $\mu \ll 1$ Eq. (4) reduces to $\mu \propto \exp(-t / t_{BB})$, while in relativistic case, $\mu \gg 1$, to $\mu \approx (t / t_{BB} + 1 / \mu_0)^{-1}$. Perhaps the most interesting is the QED ultra-relativistic Compton domain. Using Eq. (1) and asymptotic behavior of $\sigma_{\text{SN}}(\omega)$ at $\omega \to \infty$, see above, one finds out that at $\mu \gg \theta^{-1}$,

$$f \propto -\theta^3 [\ln(\theta\mu) + O(1)] \mu.$$ 

(5)

For the entire momentum span, $\mu \in (0, \mu_{p1})$, where $\mu_{p1} = k_B T_{p1} / m_0 c^2 \approx 2.4 \times 10^{22}$ is the highest momentum in the Universe corresponding to the Planck temperature $T_{p1} \approx 1.417 \times 10^{32}\text{K}$. $f(\mu)$ is numerically evaluated via Eq. (1) and depicted in Fig. 1, curve 1. Using these numerical results, it would be greatly beneficial for further analysis to have a good analytic interpolation of Eqs. (3) and (5). For $\theta \ll 1$, this task is perfectly served by a remarkably simple formula good for the entire span $\mu \in (0, \mu_{p1})$:

$$\frac{d\mu}{d\tau} = f(\mu, \theta) = -\frac{\theta^3}{q} \ln \left(1 + \frac{\mu}{\mu_C}\right) \gamma; \quad \mu_C = \frac{1}{\theta q}$$

(6)

where $q \approx 10$ is a fitting parameter, and $\mu_C$ is a “Compton threshold”; for current $\theta$ it corresponds to the energy $E_C \sim 10^{14}\text{eV}$. Eq. (6) makes better than a few percents fit to the numerics over the entire span of momentum but a small area near $\mu = \mu_C$, Fig. 1, where they are still very close. In the limit $\mu \ll \mu_C$, Eq. (6) is consistent with Eq. (3) rewritten now as $f = -\theta^3 \mu_0 \gamma$, and at $\mu \gg \mu_C$ with Eq. (5). We verified that up to relatively “hot” temperature $\theta = 0.1$, Eq. (6) remains good for other $\theta$’s too. The force $f(\mu, \theta) vs$ momentum $\mu$ for a few $\theta$’s, from current “cold” epoch, $\theta \approx 0.54 \times 10^{-9}$, to $\theta = 1$, as well as the evolution of $\mu_C$, are shown in the inset in Fig. 1.

While Eq. (6) covers the entire span of momentum, it is not yet conductive for analytic calculations of the dynamics of momentum of electrons or kinetics of density distribution, see below. Since $f = d\mu / d\tau$, the key function here is $\tau(\mu) = \int d\mu / f(\mu, \theta)$ (for a fixed $\theta$), and the conditions on a “good” function $f(\mu, \theta)$ are (a) analytic integrability of $\int d\mu / f(\mu, \theta)$, and (b) explicit “reversability” of resulting functions $\tau(\mu) \leftrightarrow \mu(\tau)$. In the Thompson domain, $\mu < \mu_C$, Eq. (3) satisfies these conditions and is fully solvable, Eq. (4). But to cover both its largest part, and at the same time – the entire immensely larger, Compton domain $\mu \in (\mu_C, \mu_{p1})$, another extremely use-

![FIG. 1: Normalized to $F_C$ light pressure on an electron by relic radiation, $|f|$, vs the normalized electron momentum, $\mu$. Curves: 1 - numerical integration of Eq. (1); 2 - Eq. (6). 3 - Eq. (7). Inset: force $|f|$, vs $\mu$ for different normalized temperatures, $\theta$. Diamonds mark Compton threshold, $\mu_C.$](image-url)
ful interpolation model was found by us as

\[ f_M(\mu, \theta) = -\left(\frac{\theta^3}{q}\right) g(1 + \ln y) \left\{ 1 + [\ln(1 + \ln y)]^{-2} \right\}^{-1} \tag{7} \]

where \( y = 1 + \mu/\mu_c \). The function \( f_M(\mu) \) for fixed \( \theta \) of current epoch is shown in Fig. 1, from \( \mu \approx 7 \) and up, the difference between (6) and (7) gets less than 1%, and then rapidly decreases. At \( \mu \ll \mu_c \), Eq. (7) yields \( f_M \approx -\theta^4 \mu^2 \), which is consistent with Eq. (3) at \( \mu \gg 1 \). Thus it is good only for a relativistic case (i. e. even for \( 1 \ll \mu \ll \mu_c \)), which is more than enough if \( \theta \ll 1 \) by insuring that momentum decay can be continually traced from far Compton \( \mu \gg \mu_c \) to low Thompson domains, \( \mu \ll \mu_c \). If necessary, the calculations for electron decay at \( \mu \ll \mu_c \) can be picked up by Eq. (3), rewritten as \( f = -\theta^4 \mu^2 \gamma \). Thus Eqs. (3) and (7) smoothly cover the entire span \( \mu \in (0, \mu_c) \), as their areas of validity overlap by orders of magnitude in \( \mu \) if \( \theta \ll 1 \). The momentum decay from initial \( \mu = \mu_o \) at \( \tau = 0 \), via integration of equation \( d\mu/d\tau = f_M \), Eq. (7), with \( \theta = \text{const} \), is

\[ \frac{\mu(\tau)}{\mu_c} = \exp \left\{ \exp \left[ \frac{(\tilde{\tau}_0 + \tilde{\tau})^2}{4} - 1 \right] \right\} \tag{8} \]

where \( \tilde{\tau} = (\theta^3/\gamma)\tau \) and

\[ \tilde{\tau}_0 = \chi^{-1} - \chi \quad \text{with} \quad \chi = \ln[1 + \ln(1 + \mu_o/\mu_c)] \tag{9} \]

For \( \mu_o/\mu_c = 2 \times 10^5 \) (or initial energy \( E_o \approx 2 \times 10^{19} \text{eV} \) which is slightly below the highest particle energy, \( E_0 \approx 5 \times 10^{19} \text{eV} \), observed in cosmic rays [23]), \( \mu(\tau) \) is depicted in Fig. 2, curve 1. The time \( \int_{\mu_c}^{\mu} d\mu/[f_M] \) for an electron to lose about \( 2 \times 10^6 \) of its momentum during the “Compton phase” \( \mu_o \to \mu_c \) is \( \Delta t_{\mu_c} \approx 2.57 \), hence \( (\Delta t_c)_{\mu_c} \approx 5.3 \times 10^{11} \text{s} \), whereas immediately after that, within the same time period, \( \mu \) loses much less than a factor of magnitude. The above calculated transition time is realistically the longest possible for the observed sources for an initial energy lower than \( 5 \times 10^{19} \text{eV} \), this time will be even shorter. Following that, \( \mu \) decays even slower as \( \approx 1/t \) [see Eq. (4) and text below it]. The explanation of this dramatic difference (see e. g. [14,19]) is that while the cross-section of photon scattering by electrons is slowly receding with energy, each act of ICS gets much more “quantum efficient” as \( \mu \) exceeds \( \mu_c \).

In ICS, a low-energy photon scattered from a high-energy electron gets a huge boost by accruing up to almost full energy of the electron. The peak gain is reached in a “head-on” collision, when a photon is exactly back-scattered. Based on the Compton scattering formula [20], Eq. (8), the maximum scattered photon energy in the L-frame is \( \epsilon_{sc} \equiv \hbar \omega_{sc}/m_0 c^2 = \epsilon_{in}(\gamma + \mu)/(\gamma - \mu + 2\epsilon_{in}) \) where \( \epsilon_{in} \) is the incident photon energy. For high-energy photons, \( \gamma \approx \mu \gg 1 \), and low-energy incident photons, \( \epsilon_{in} \ll 1 \), the maximum quantum efficiency of ICS defined as the ratio of scattered photon energy to that of an incident electron, \( \eta \equiv \epsilon_{sc}/\gamma \), is then \( \eta \approx 1 + (4\epsilon_{in})^{-1} \), and in the sub-QED domain, \( \gamma \ll 1/\epsilon_{in} \), we have \( \eta \approx 4\gamma \epsilon_{in} \ll 1 \), hence small loss of electron energy per collision. However, in QED Compton domain, \( 4\gamma \epsilon_{in} \gg 1 \), we have \( \eta \approx 1 - (4\epsilon_{in})^{-1} \sim 1 \), i. e. an electron passes great part of its energy to a scattered photon. With considerable probability an electron jumps then in one collision to the Compton threshold, \( \mu \to \mu_{sc} \sim \mu_c \).

As \( \mu \) keeps decaying from \( \mu_c \) down to a relativistic threshold, \( \mu = 1 \), its dynamics slows down tremendously. The transition time here is \( \Delta \tilde{\tau}_{Th} = (\theta^2/q) \int_{\mu_{sc}}^{\mu_c} d\mu/[f_M] \approx \mu_c = 1.87 \times 10^8 \), hence \( \Delta t_{\mu_c} \approx 4 \times 10^{19} \text{s} \), i. e. 8 orders of magnitude longer than \( \Delta t_{\mu_c} \), and 2 orders – of the current age of Universe, \( t_u \). Further decay to a thermal equilibrium, \( \mu = \theta \), slows down even stronger: since in Eq. (4) \( t_{BB} \approx 4 \times 10^{19} \text{s} \), we have \( \Delta t_{BB} \approx 0.85 \times 10^{27} \text{s} \). All this makes a relaxation in the Thomson domain, \( \mu \ll \mu_{sc} \), completely negligible and leads to the formation of a quasi-stable phase, a “frozen non-equilibrium” at lower \( \mu \). Its current spectral distribution could be obtained by solving a Fokker-Planck equation beginning perhaps from leptogenesis, which is outside of the scope of this Letter; most likely its most active formative period was the epoch near \( \theta = q^{-1/2} \).

It is obvious, however, that the electron spectrum and its cutoff are shaped by the relic radiation, to a degree similar to the cutoff for protons [16], making the “imprint” of low-energy CMB onto high-energy electron spectrum far below the current Compton threshold \( E_c \approx 10^{14} \text{eV} \). Assuming that (a) the electron distribution is isotropic, same as CMB, (b) the total number of electrons is \( \approx \text{const} \), \( \int p^{(c)} dp = \text{inv} = N_c \), where \( p^{(c)}(\mu, \tau) = 4\pi \mu^2 \ g^{(c)}(\mu, \tau) \) is the density distribution of \( \mu \), and (c) the thermal equilibrium of a relativistic gas at any \( \theta \) is due to the Maxwell-Jüttner (MJ) distribution [24], \( g^{(c)}(\mu, \tau) \propto e^{-\gamma/\theta} \) (a generalization of Maxwell-Boltzmann distribution), we found a Fokker-Planck equation for the distribution function \( g^{(c)}(\mu, \tau) \), in terms of dimensionless...
momentum $\mu$, time $\tau$, temperature $\theta$, and force $f$, as
\[
\frac{\partial [\mu^2 g^{(e)}]}{\partial \tau} + \frac{\partial}{\partial \mu} \left\{ \mu^2 f(\mu, \tau) \left[ g^{(e)} + \theta(\tau) \gamma \frac{\partial g^{(e)}}{\partial \mu} \right] \right\} = 0
\]
Integration of Eq. (10) over the span of tens orders of magnitude of those factors will be addressed by us elsewhere. Yet a great deal of insight can be gained for the case $\theta \approx const \ll 1$ and $\gamma \sim \mu \gg 1$, which is of major interest. Since in such a case the system remains extremely far from the equilibrium during the evolution, see above, the last term in Eq. (10) can be omitted, and in terms of density $\rho^{(e)} \propto \mu^2 g^{(e)}$ it can be reduced to
\[
\frac{\partial \rho^{(e)}}{\partial \tau} + \partial \left[ f \rho^{(e)} \right]/\partial \mu = 0
\]
which is essentially a continuity-like equation. Luckily, it is fully integrable; its general solution is
\[
\rho^{(e)} = \Phi(\xi - \tau)/f(\mu), \quad with \quad \xi = \int d\mu/f
\]
where $\Phi(s)$ is an arbitrary function of $s$ defined by initial conditions, e.g. the MJ-distribution with $\theta_0 = \Theta \gg 1$. A resulting analytic solution for $\rho^{(e)}(\mu, \tau)$ with $f = f_M$, Eq. (7), for $\rho^{(e)} vs \mu$ for various times $\tau = (\theta^3/q)\tau$ is plotted in Fig. 3 for initial temperature, $\Theta = 10^5 \mu_c$, or $k_B T_0 = 10^{19} V$. A curve at $\tau = 0$ there depicts an initial MJ-distribution, $\rho^{(e)}(\mu) \propto \mu^2 e^{-\gamma/\theta}$, which peaks at $\mu = 2\Theta$, i.e. $E_0 = 2 \times 10^{19} V$, same as for a single electron example, see the text following Eq. (9). The transient distribution peak moves fast in the beginning and then slows down tremendously as it reaches $\mu = \mu_c(\theta)$, after which it moves as $\tau^{-1}$. The evolution of its position, $\mu_{pk}$, coincides with the timeline of a single electron with $\mu_0 = 2\Theta$, see curve 1 in Fig. 2, whereas its intensity $\rho_{pk}(\tau)$, curve 2 in Fig. 2, goes up orders of magnitude higher than that of the initial MJ-distribution; at $\mu < \mu_c$, $\rho_{pk} \propto \tau^2$. Its width narrows down respectively, $\Delta \mu/\mu_{pk} \propto \tau^{-2}$ so that for e.g. $\mu = 20 \ (E = 10 MeV)$, it reaches $\Delta E \sim 2 KeV$ i.e. $\Delta \mu/\mu_{pk} \sim 2 \times 10^{-4}$, compared to the initial relative width $\Delta \mu_0/\mu_0 \sim 1.4$. This “implosion shock” or collapse is due to a “pile-up” effect with the leading downward front moving slower than a trailing, higher front, resulting in an unexpected almost mono-energetic line. The effect is reminiscent of the shock precursor formation in the Coulomb explosion of charged particles [25] and in astrophysics [26], although a breaking point here is never reached. The narrow lines would indicate signals from a very far and hot source, hence its importance. It may necessitate the development of spectral techniques for their detection. Those lines might be broadened up by various “local” factors, which calls for detailed consideration of how they may affect the formation of those lines in future research.

In conclusion, we developed an analytic theory for the light pressure on electrons by relic radiation that covers the entire span of energies/momenta up to the Planck energy, and analyzed the kinetics of electron relaxation from the highest to lowest energies. We used relativistic Fokker-Planck equation to explore the evolution of spectra of high-$T$ sources and predicted their implosion into very narrow lines due to pile-up effect. We pointed out that a very low rate of electron energy decay in the Thompson domain may result in the formation of frozen non-equilibrium state forming an imprint by low-$T$ relic radiation “projected” into relativistic distribution.

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[20] See Supplemental Material at [URL will be inserted by publisher] for derivation of Eq. (1), the details of Klein-Nishina formula for $\sigma_K$, cross-section-like term $\sigma_R$, and Compton scattering formula.

[21] An alternative formula can be obtained from Eq. (1) by changing the order of integration, [20], Eq. (17).


Compton light pressure and spectral imprint of relic radiation on cosmic electrons.
(Supplemental Material)

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LORENTZ TRANSFORMATION OF RADIATION SPECTRUM

We develop here a relativistic/QED theory of a radiation pressure acting upon a charged particle (electron) in an isotropic and spatially homogeneous radiation with an arbitrary spectrum. An isotropic radiation is necessarily associated with a certain preferred, L-frame, so that a particle at rest in that frame experiences no time averaged radiation pressure, as the action of a $\vec{k}$-vector radiation component is canceled by a counter-propagating ($-\vec{k}$) component.

While considering a relativistic interaction of a particle with a radiation, we use a direct Lorentz transformation of the radiation spectrum \[1,2\] instead of a conventional approach based on 4D stress-tensor of the field. In its final form it produces a radiation pressure for an arbitrary spectral density of radiation, $\rho(\omega)$, arbitrary frequency-dependence $\sigma(\omega)$ of particle cross-section $\sigma$, and arbitrary relativistic factor $\gamma$. It coincides with the well known result in the limiting case $\sigma = $ const, in particular in the case of the so called Thompson scattering, yet readily allows one to take into consideration the inevitable frequency dependance of $\sigma(\omega)$ (in particular, the Klein-Nishina electron cross-section) by accounting for the relativistic Doppler shift of radiation spectrum seen by the particle.

For a gas of non-zero mass particles one can define distribution function $g(\vec{p}, \vec{r})$ in a certain (lab) L-frame in the phase space of momentum $\vec{p}$ and position vector $\vec{r}$ as the number of particles, $dN_{\vec{p},\vec{r}} = g(\vec{p}, \vec{r})d\Omega$, per the element of phase space, $d\Omega = dV_p dV_r$, where $dV_p = dp_x dp_y dp_z$ and $dV_r = dx dy dz$ are the elements of momentum and coordinate spaces respectively. A general formula \[1,2\] for a Lorentz transformation of a distribution function $g_L(\vec{p}_L, \vec{r}_L)$ in a L-frame to a distribution function $g_p(\vec{p}_P, \vec{r}_P)$ in any P-frame moving uniformly with respect to the L-frame is as:

$$g_L(\vec{p}_L, \vec{r}_L) = g_P(\vec{p}_P, \vec{r}_P); \quad d\Omega_L = d\Omega_P$$

(1)

where $\vec{k}_L$ and $\vec{r}_L$ are related to $\vec{k}_P$ and $\vec{r}_P$ respectively by a standard Lorentz transform for an observer moving in L-frame in the x-axis with velocity $\vec{v} = \vec{e}_x v$. In the case of photon gas Eq. (1) remains true for the spectrum of photons $g(\vec{k}, \vec{r})$, by replacing $\vec{p}$ with $\vec{k}$ ($k = \omega/c$), $dV_p$ by $dV_k = dk_x dk_y dk_z$, and $d\Omega$ by $d\tilde{\Omega} = dV_k dV_r$. Since we are interested here in the case of a homogeneous radiation, a spectrum is $g(\vec{k})$, and its transformation is written as

$$g_L(\vec{k}_L) = g_P(\vec{k}_P), \quad d\tilde{\Omega}_L = d\tilde{\Omega}_P; \quad \text{with } g \text{ and } d\tilde{\Omega} \text{ - dimensionless}$$

(2)

where Lorentz transform for $\vec{k}$ is $(k_x)_L = \gamma(k_x)_P + \beta k_y; \quad (k_y)_L = (\vec{k}_L)_P; \quad k_z = \gamma[k_z + \beta (k_x)_P]$, where $k_{\perp} = k_y e_y + k_z e_z, \quad k = \omega/c = \sqrt{k_x^2 + k_y^2}, \quad \beta = v/c, \quad \text{and } \gamma = 1/\sqrt{1 - \beta^2}$. In particular, the Doppler coefficient is as

$$D \equiv \omega_P/\omega_L = [\gamma(1 + \beta \cos \xi_p)]^{-1} = \gamma(1 - \beta \cos \xi_L)$$

(3)

where $\xi_{(\perp)}$ are the angles between respective $\vec{k}$-vectors and the x-axis, i.e. $\cos \xi_L = (k_x)_L/k_z, \cos \xi_P = (k_x)_P/k_z$, and are transformed as $\cos \xi_L = (\cos \xi_L - \beta)/(1 - \beta \cos \xi_L)$. Furthermore, since the radiation is isotropic in the preferred L-frame, the distribution function $g_L$ does not depend on the direction of $\vec{k}_L$-vector, and we have $g_L(\vec{k}_L) = g_L(k_z)$, where $k = |\vec{k}|$. Since both spectra are symmetrical around the x-axis, it is convenient to use spherical coordinates in the $\vec{k}$ space, so that $dV_k = k^2 dk dO$, where $dO$ is the element of solid angle in the direction of $\vec{k}$. One can then introduce another spectrum, $\rho(\vec{k})$, which describes the number of photons in the element $dk dO V_\vec{k}$ and is defined as

$$\rho(\vec{k}) = k^2 g(\vec{k}) \quad \text{with } \quad dN_{\vec{k}} = \rho(\vec{k}) dk dO dV_\vec{k}; \quad [\rho] = \text{cm}^{-2}$$

(4)

whose transformation using Eq. (2) is written as:

$$\rho_L(k_z)/k_z^2 = \rho_P(\vec{k}_P)/k_z^2$$

(5)

From now on, since we are dealing with a homogeneous radiation independent on the radius-vector length, $|\vec{r}|$, we will re-assign the notion of spectra $\rho$ only to the ones being functions of frequency $\omega$ and angles $\xi$ instead of $\vec{k}$ and $\vec{r}$ vectors. In this case, $\rho(\omega, \xi) d\omega$ will have dimension of $[\text{cm}^{-3}]$. As expected, the radiation becomes anisotropic in the P-frame (yet symmetric around the x-axis) with its spectrum $\rho_p(\omega; \xi_p)$ given by an L-frame isotropic spectrum.
function $\rho_\perp(\omega)$, whose argument and amplitude are now altered by the parameters $\beta$ and $\xi_\perp$ \textit{via} Doppler coefficient $D = \left[\gamma(1 + \beta \cos \xi_\perp)\right]^{-1}$, Eq. (3):

$$\rho_\perp(\omega; \xi_\perp) = \rho_\perp[\omega/D(\beta, \xi_\perp)] \times D^2(\beta, \xi_\perp)$$  \hspace{1cm} (6)

Our further calculations will be focussed on the events in the $P$-frame assuming that the spectrum $\rho_\perp(\omega)$ is known.

\textbf{PHOTON SCATTERING AND MOMENTUM TRANSFER}

In quantum terms, the incident wave function is a plane-wave with amplitude 1, $\Psi_{\text{sc}} = \text{exp}(\vec{k}_{\text{in}} \cdot \vec{r})$, the scattered one is $\Psi_{\text{sc}} = \Phi(\vec{k}/\kappa)/r$, and a differential cross-section is $d\sigma/dO = |\Phi|^2$, which is the probability of finding a scattered particle within a solid angle $dO$. We designate the wave-vector of an incident photon in the $P$-frame as $\vec{k}_{\text{in}}$, and that of a scattered photon as $\vec{k}_{\text{sc}}$. The latter one is scattered into the solid angle $dO_{\text{sc}} = \sin \psi_{\text{sc}} d\psi_{\text{sc}} d\phi$ around $\vec{k}_{\text{sc}}$. Here $\psi_{\text{sc}} \in [0, \pi]$ is the angle between incident and scattered $k$-vectors, and $\phi \in [0, 2\pi]$ is an azimuthal angle around the $\vec{k}_{\text{sc}}$ direction. The number of photons scattered into $dO_{\text{sc}}$ within time interval $dt$ and spectral band $d\omega = c \times dk$, is

$$dN_{\text{sc}} = \rho_{\perp}(\omega_{\text{in}}, \xi_{\text{in}}) \frac{d\sigma}{dO_{\text{sc}}} \frac{dO_{\text{sc}}}{4\pi} c dt d\omega$$  \hspace{1cm} (7)

The Compton quantum scattering formula determines the ratio $R$ of photon energies after and before scattering:

$$\epsilon_{\text{sc}}/\epsilon_{\text{in}} = \omega_{\text{sc}}/\omega_{\text{in}} \equiv R(\epsilon_{\text{in}}, \psi_{\text{sc}}) = [1 + \epsilon_{\text{in}}(1 - \cos \psi_{\text{sc}})]^{-1} \text{ with } \epsilon \equiv h\omega/m_0 c^2$$  \hspace{1cm} (8)

When calculated back to the $L$-frame, photons back-scattered from an ultra-relativistic electron, $\gamma \gg 1$, may have large energies with the Doppler shift up to $D \approx 2\gamma$, even if their energies in the $P$-frame are still below QED limit, i.e. $\epsilon_{\text{in}} \ll 1$. In the astrophysical context, it is called an Inverse Compton (or Thompson, if $\epsilon_{\text{in}} \ll 1$) scattering. It is common to distinguish between ICS from relic radiation (which is the subject of this paper) and the so called Sunyaev-Zel’dovich effect [3] regarded as \textit{localized} phenomenon seen towards clusters of galaxies [see e.g. review [4]]. In addition to the radiation pressure, this scattering may also result in a diffusion of electrons [5].

Only the projection of $\vec{k}_{\text{sc}}$ vector into $\vec{k}_{\text{in}}$-direction will contribute to the force $F$; all the rest will be canceled out after integrating over the azimuthal angle $\phi$ (in the $P$-frame, where the electron is at rest, the scattering problem has a symmetry around $\vec{k}_{\text{in}}$). Thus the momentum transfer to an electron in $\vec{k}_{\text{in}}$-direction after the scattering is

$$\hbar \Delta k_{\text{sc}} = h(k_{\text{in}} - k_{\text{sc}} \cos \psi_{\text{sc}}) = h\omega_{\text{in}} [1 - R(\epsilon, \psi_{\text{sc}}) \cos \psi_{\text{sc}}]/c$$  \hspace{1cm} (9)

\textbf{RADIATION PRESSURE ON A CHARGED PARTICLE}

Considering the number of photons, $dN_{\text{sc}}$, Eq. (7), scattered into a solid angle $dO_{\text{sc}}$, the radiation pressure impacted by them on the electron in $\vec{k}_{\text{in}}$-direction, is the rate of momentum transfer, $f = f\vec{q}_{\text{in}}$ where $\vec{q}_{\text{in}} = \vec{k}_{\text{in}}/k_{\text{in}}$ and

$$df = h\Delta k_{\text{sc}} \frac{d^2N_{\text{sc}}}{dk_{\text{in}} d\omega_{\text{in}}} = \frac{h\omega_{\text{in}}}{4\pi} \frac{d\sigma}{dO_{\text{sc}}} [1 - R(\epsilon, \psi_{\text{sc}}) \cos \psi_{\text{sc}}] \rho_{\perp}(\omega_{\text{in}}, \xi_{\text{in}}) \sin \psi_{\text{sc}} d\psi_{\text{sc}} d\phi$$  \hspace{1cm} (10)

Integrating Eq. (10) over the angles $\phi$ and $\psi_{\text{sc}}$, we find a full $\omega_{\text{in}}$-Fourier component of the radiation pressure as:

$$f(\omega_{\text{in}}, \beta, \bar{q}_{\text{in}}) = \frac{h\omega_{\text{in}}}{4\pi} \rho_{\perp}(\omega_{\text{in}}, \xi_{\text{in}}) \sigma_{\text{MT}} \text{ with } \sigma_{\text{MT}} = 2\pi \int_0^\pi [1 - R(\epsilon, \psi_{\text{sc}}) \cos \psi_{\text{sc}}] \frac{d\sigma}{dO_{\text{sc}}} \sin \psi_{\text{sc}} d\psi_{\text{sc}}$$  \hspace{1cm} (11)

where we introduced the notion of a cross-section $\sigma_{\text{MT}}$ describing a momentum transfer from $\omega_{\text{in}}$-photons to a particle. It must be noted that $\sigma_{\text{MT}}$ does \textit{not} in general coincide with a plain \textit{full} (integrated) scattering cross-section, $\sigma(\epsilon_{\text{in}})$ [which in the case of an electron is determined by the so called Klein-Nishina formula, see below, Eq. (20)]:

$$\sigma(\epsilon_{\text{in}}) = 2\pi \int_0^\pi \frac{d\sigma(\epsilon_{\text{in}})}{dO_{\text{sc}}} \sin \psi_{\text{sc}} d\psi_{\text{sc}} \text{ with } \epsilon_{\text{in}} \equiv \frac{h\omega_{\text{in}}}{m_0 c^2}$$  \hspace{1cm} (12)

simply because only the projection of $\vec{k}_{\text{sc}}$ into $\vec{k}_{\text{in}}$-direction contributes to the radiation force $F$, see Eqs. (9) and (11), where in general $R(\epsilon, \psi_{\text{sc}}) \neq 0$ (if $\epsilon > 0$). The relation between $\sigma_{\text{MT}}$ and $\sigma(\epsilon_{\text{in}})$ is as:

$$\sigma_{\text{MT}} = \sigma - \sigma_R \text{ with } \sigma_R(\epsilon) = 2\pi \int_0^\pi R(\epsilon, \psi) \frac{d\sigma}{dO} \cos \psi \sin \psi d\psi = 2\pi \int_{-1}^1 \frac{d\sigma}{dO} \frac{\zeta d\zeta}{1 + \epsilon(1 - \zeta)}$$  \hspace{1cm} (13)
where we introduced a cross-section-like term $\sigma_{\mu}$, that reflects the projection-related $R$-factor in Eq. (9) and (11). This term zeroes out for $\sigma = \text{const}$ (e.g. Thompson scattering) and at $\epsilon = 0$; it peaks out at $\epsilon \sim 0.54$ (see the end of last Section), but is negligible small both at $\epsilon \ll 1$ and $\epsilon \gg 1$. Note also that only full (integrated) cross-sections, Eqs. (12),(13), enter into radiation pressure calculations.

Now, we compute the radiation pressure, $F = F(p)\hat{e}_x$, as an integral of the projections of the Fourier force components, $f(\omega_{in}, \beta, \xi_{in})$ into the $x$-axis, i.e. $f_x = \int f(\omega_{in}, \beta, \xi_{in})\hat{e}_x = f\cos\xi_{in}$, over all the incident solid angles, $dO_{in} = \sin\xi_{in}d\xi_{in}d\phi_{in}$, and frequencies $\omega_{in}$ in the $P$-frame. Thus, we have for the full radiation pressure

$$F(p) \equiv \frac{dp}{dt} = \int \int \frac{df_x}{dO_{in}}dO_{in}d\omega = 2\pi \int_{0}^{\pi} \left[ \int_{0}^{\pi} f(\omega_{in}, \beta, \xi_{in}) \sin\xi_{in} \cos\xi_{in} d\xi_{in} \right] d\omega \tag{14}$$

where $p = \beta \gamma mc^2$ is the electron momentum. Recalling that a spectrum $\rho_\gamma$ in $f$, Eqs. (12)-(14) in the $P$-frame can be expressed directly via the known isotropic spectrum in the $L$-frame, $\rho_{L}$, Eq. (6), we can now write the expression for the radiation pressure in a closed form as:

$$F(p) = \frac{\hbar}{4} \int_{0}^{\pi} \omega \sigma_{\mu\tau}(\omega) \left[ \int_{0}^{\pi} \sin(2\xi) \rho_{L}(\omega/D)D^2d\xi \right] d\omega \tag{15}$$

or by using a substitution, $\zeta = \cos\xi$, reduce it to:

$$F(p) = \hbar \int_{0}^{\pi} \frac{\omega \sigma_{\mu\tau}(\omega)}{2\gamma^2} \left[ \int_{-1}^{1} \rho_{L} \frac{(1 + \beta \zeta)}{(1 + \beta^2)^{3/2}} \zeta d\zeta \right] d\omega \tag{16}$$

Alternatively, by using a substitute $\nu = \omega(1 + \beta \zeta)$, the same result can be written as

$$F(p) = \hbar \int_{0}^{\pi} \frac{\nu \rho_{L}(\nu)}{2\gamma^4} \left[ \int_{-1}^{1} \sigma_{\mu\tau} \left( \frac{\nu}{1 + \beta \zeta} \right) \zeta d\zeta \right] d\nu \tag{17}$$

Which one of Eqs. (16) or (17) to use for detailed study is a matter of computational convenience depending on specific model functions $\sigma_{\mu\tau}(s)$ and $\rho_{L}(\nu)$ [6].

**QED Scattering and Momentum-Transfer Cross-Sections for Electron**

Eqs. (16), (17) allow for full accounting of the cross-section $\sigma$ dependance on frequency/energy of the photon $\epsilon = h\omega/m_ec^2 \ll 1$. In a low-frequency limit, or non-relativistic case, $\epsilon \ll 1$, the scattering by a charged particle can be described by a frequency-independent Thompson cross-section

$$\sigma = \text{const} = \sigma_o = (8\pi/3)r_o^2; \quad \text{with} \quad r_o = e^2/m_ec^2 \tag{18}$$

where $r_o$ is a classical EM-radius of a particle. Yet a cross-section becomes frequency-dependent even at sub-relativistic frequencies, which in the case of electrons/leptons is due to quantum Compton scattering via Klein-Nishina formula (see below). Let us start with the differential cross-section, $d\sigma_{KN}/dO$ (we still need it for computing first the differential Compton cross-section $d\sigma_c/dO$, Eq. (13)). We have from [7,8]:

$$d\sigma_{KN}/dO = (3\sigma_o/16\pi)R^2 (R + R^{-1} - \sin^2\psi) \tag{19}$$

with $R = R(\epsilon, \psi)$ as in Eq. (8). Using Eq. (19) in Eq. (12) we get a well-known full Klein-Nishina cross-section:

$$\frac{\sigma_{KN}(\epsilon)}{\sigma_o} = \frac{3}{8\epsilon} \left[ \left( 1 - \frac{2}{\epsilon} - \frac{2}{\epsilon^2} \right) \ln(1 + 2\epsilon) + \frac{1}{2} + \frac{4}{\epsilon} - \frac{1}{2(1 + 2\epsilon)^2} \right] \quad \text{with} \quad \epsilon \equiv \frac{h\omega}{m_ec^2} \tag{20}$$

In the limits of “cold” and “hot” radiation we have respectively, $\sigma_{KN}/\sigma_o \approx 1 - 2\epsilon$ at $\epsilon \ll 1$; and $\sigma_{KN}/\sigma_o \approx (3/8\epsilon)[\ln(2\epsilon) + 1/2]$ at $\epsilon \gg 1$. Finally, using Eq. (8) in Eq. (13), we have

$$\frac{\sigma_n}{\sigma_o} = \frac{3}{8} \int_{-1}^{1} \tilde{R}^2[\tilde{R}^2 - (1 - \zeta^2)\tilde{R} + 1][\zeta d\zeta] \quad \text{with} \quad \tilde{R} = \frac{1}{1 + \epsilon(1 - \zeta)} \tag{21}$$

the evaluation of integral in which yields:

$$\frac{\sigma_n(\epsilon)}{\sigma_o} = \frac{3}{8} \left\{ \frac{2(1 + 2\epsilon)}{3(1 + 2\epsilon)^3} + \left\{ \frac{2(1 + \epsilon)}{\epsilon(1 + 2\epsilon)} \right\} \ln(1 + 2\epsilon) + \frac{2(2 + 4\epsilon + \epsilon^2)}{\epsilon(1 + 2\epsilon)^2} - \frac{3(1 + \epsilon)}{\epsilon^3} \left[ \frac{2(1 + \epsilon)}{1 + 2\epsilon} - \frac{\ln(1 + 2\epsilon)}{\epsilon} \right] \right\} \tag{22}$$
As mentioned already, $\sigma_R$ zeroes out at $\epsilon = 0$. The three terms within the “outer” brackets $\ldots$ in Eq. (22) are grouped in such a way as to have each one of them to also zero out at $\epsilon = 0$. In the limits of cold and hot radiation we have respectively $\sigma_R/\sigma_0 \approx 6\epsilon/5$ at $\epsilon \ll 1$, and $\sigma_R/\sigma_0 \approx (1/2\epsilon)[1 - 3 \ln(1 + 2\epsilon)/4\epsilon]$ at $\epsilon \gg 1$, so that in rough approximation, the term $\sigma_R$ in the theory of photon–electron momentum transfer can be neglected at $\epsilon, \epsilon^{-1} \ll 1$. In general, it can be shown that $\sigma_{KN} > \sigma_R > 0$ everywhere in $\epsilon \in (0, \infty)$, so that total cross-section $\sigma_{MT} = \sigma_{KN} - \sigma_R$ in Eqs. (15) and (16) is always positively defined. Spectral profiles of all these three cross-sections are depicted at Fig. 1 showing that $\sigma_R(\epsilon)$ reaches its maximum $\sigma_R/\sigma_0 \approx 0.1442$ at $\epsilon \approx 0.543$, or $h\omega \approx 277.5$ KeV; at that point $\sigma_{KN} \approx 0.5488\sigma_0$, and $\sigma_{MT} \approx 0.4024\sigma_0$.

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Fig. 1: Dimensionless integrated cross-sections of photons scattering by an electron: Klein-Nishina $\sigma_{KN}/\sigma_0$, projection-related term $\sigma_R/\sigma_0$, and momentum-transfer $\sigma_{MT}/\sigma_0 = (\sigma_{KN} - \sigma_R)/\sigma_0$, vs dimensionless energy of incident photons, $\epsilon = h\omega/m_0c^2$.

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[6] Eqs. (16) and (17) are very general in that they are applicable to any particles, not only elementary ones; they are based only on the knowledge of $\sigma_{MT}$ and $\rho_\omega$ as functions of $\omega$. They will be valid for example for the resonant or other dispersion-related interaction of radiation with atoms or even macro-particles, with any quantum or classical resonances due to e. g. dipole momenta, band structure, eigen-modes, etc. It also does not matter whether the full cross-section $\sigma_{MT}$ is due to elastic scattering or includes losses of energy to internal degrees of freedom; all that implicitly enters into the function $\sigma_{MT}(\omega)$.

This is reminiscent of a phenomenological role played by dispersive dielectric constant $\varepsilon(\omega)$ in electrodynamics whereby $\varepsilon(\omega)$ provides a short-hand representation of all the constitutive interactions of EM-field with matter.
